Handout #3

Title: Foundations of Econometrics — Fall/2015
Course: Econ 367 — Instructor: Dr. I-Ming Chiu

MATHEMATICAL FUNDAMENTALS & PROBABILITY THEORY

MATHEMATICAL FUNDAMENTALS

1. Sets and Subsets

Definition

Set: A collection of objects. Each object is an element. How to express a set: a. enumeration method, b. property method.
e.g. (a): Econ367 = {Alia, Babu, Christa, Corey, I-Ming, Joseph, Omar, William}
e.g. (b): L (labor force) = {x: x is a person who belongs to the civilian non-institutional population at age 16 and above who has a job or search for a job}

Important Sets for Numbers

The set of natural numbers, $N = \{1, 2, 3, \ldots\}$
The set of integers, $Z = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$
The set of rational numbers, $Q = \{q: q = \frac{s}{t}; s, t \in Z \ \& \ t \neq 0\}$
The set of real numbers, $R = \{-\infty, \infty\}$

Different ways to present a set: $A = \{-2, -1, 0, 1, 2\} \equiv \{x \in Z: x^2 < 5\} \equiv \{x \in Z: |x| \leq 2\}$

Subset and Set Operation

1. Subset: $X \subseteq Y$; all the elements of a set X are also elements of set Y.
Proper subset: $X \subset Y$; all the elements of a set X are also elements of a set Y, but not all the elements of Y are in X; proper subset excludes the situation where X = Y.

2. Sample Space (S) or Universal set: the collection of all the outcomes from an experiment. Sample space can be finite or infinite
e.g. Toss a coin once.
e.g. Toss a coin till the head appears.

Event: An event is a subset of the sample space. It can be simple or compound.
e.g. The instructor in Econ367
e.g. Female students in Econ367

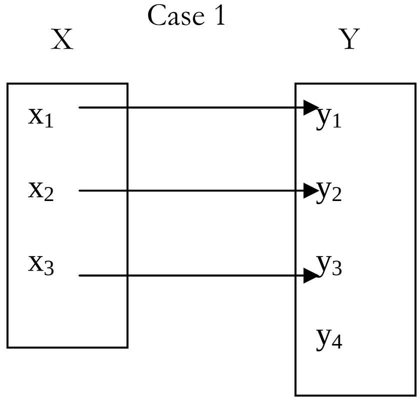
Set Operations (**Venn diagrams** can be used to illustrate these operations)

a. Intersection: $C = A \cap B = \{x: x \in A \text{ and } x \in B\}$
b. Union: $D = A \cup B = \{x: x \in A \text{ or } x \in B\}$
c. Complement: $\overline{A} = S - A = \{x \in \Omega: x \notin A\}$
d. Relative difference: $E = A - B = \{x: x \in A \text{ and } x \notin B\}$
e. Partition: $X_i \subset S$ $(i = 1\ldots n)$, $X_i \cap X_j = \phi$ $(i \neq j)$, $\cup_{i=1\ldots n} X_i = S$.
e.g. S = labor force, X_1 = employed (has a job), X_2 = unemployed (look for a job)
f. Power set of a set X: $P(X) = \{A: A \subseteq X\}$. The total number of $P(X) = 2^N$; N; the number of elements in set X. Empty set ($\varnothing$): a set does not contain any elements.

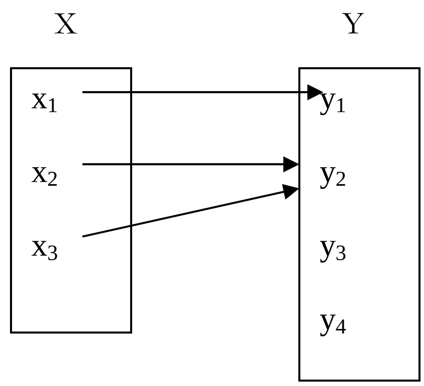
e.g.: Throw a dice once
$S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 3, 5\}$, $B = \{4, 5, 6\}$
1. $A \cup B$ =? Answer: $\{1, 3, 4, 5, 6\}$
2. $A \cap B$ =? Answer: $\{5\}$ **if $A \cap B = \varnothing$ we say A and B are disjoint (mutually exclusive)
3. $\overline{A}$ =? Answer: $\{2, 4, 6\}$
4. A - B =? Answer: $\{1, 3\}$ Notice that A - B is equivalent to $A - (A \cap B)$
5. $(A \cup B) - (A \cap B)$ =? Answer: $\{1, 3, 4, 6\}$

2. Functions

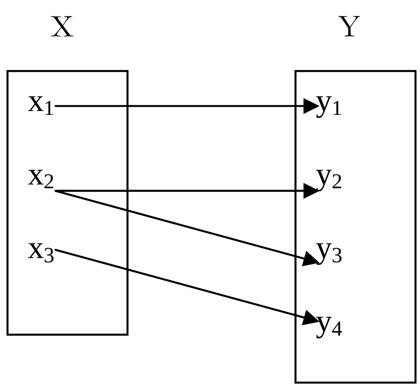
Case 1



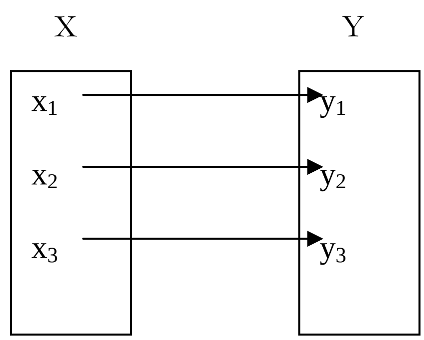
Case 2



Case 3



Case 4



Function

Def: A function from set X to set Y is a rule that associates with each element of X, one and only one, element of Y.

Function is an association.

$f: X \rightarrow Y$

or $y = f(x), x \in X$

X: Domain

Y: Codomain

Range: the set of elements in Y; that is $\{y_1, y_2, y_3\}$ in case 1 from the above graphical

e.g. X = {lower GDP, higher GDP}, Y = {contractionary policy, expansionary policy},
X = {x: x = study time in # of hours}, Y = {y: y = exam scores}

Function is a mapping.

The range of a function can be written as the Image set:

$f(X) = \{y \in Y: y = f(x), x \in X\}$

e.g. for case 1 & 4, the image set is $\{y_1, y_2, y_3\}$.

$\{y_1, y_2\}$ is the image set for case 2.

Function: a) one-to-one: Case 1
b) Several-to-one: Case 2

Function: a) Maps onto: $f(X) = Y$, Case 4
b) Maps into: $f(X) \subset Y$, Case 1, 2.

The inverse of a function exists iff (i.e., if and only if) the function is one-to-one and maps onto.

3. Counting

Why do we have to count?

Multiplication Principle

e.g. There are three high ways from Boston to NYC and there are two high ways from NYC to Philadelphia, how may combination of routes someone can take traveling between Boston and Philadelphia?

Factorial (n!): $n! = n*(n-1)*(n-2)*...*2*1$

Note: $0! = 1$, $n! = n*(n-1)!$

Permutation: The number of permutations (ordered choices) of k objects from n objects is

$$P_{k,n} = n*(n-1)*\ldots*(n-k+1) = \frac{n!}{(n-k)!}$$

Combination: The number of combinations (unordered choices) of k objects from n objects is

$$C_{k,\,n} = \frac{n!}{k!(n-k)!}$$; the other common notation for combination is $\binom{n}{k}$

Note: $C_{k,\,n} = \frac{P_{k,n}}{k!}$, what is the intuition behind this equality? (pay attention to the divisor: r!)

e.g. A committee that consists of president, vice president, and treasury will be formed from a class of ten students, how many different committees can be formed?
Answer: $P_{3,\,10}$
e.g. Three students are needed to form a committee from a class of ten students, how many different committees can be formed?
Answer: $C_{3,\,10}$
e.g. What's the odd (probability) to win the grand prize of Powerball lottery?
Answer: $\frac{1}{\binom{59}{5}\binom{35}{1}}$

PROBABILITY THEORY

1. Interpretation of Probability
What's probability?
Answer: A numerical description of uncertainty.

Two schools: Frequentist vs. Bayesian

2. Axioms of Probability
Three components in a probability space according to Russian mathematician A. N. Kolmogorov:

(a) A Sample Space
(b) Events
(c) A Probability Measure

The collection of events must satisfy the following property, called σ-field:

(a) The sample space is an event
(b) If E is an event, then E^C is an event
(c) The union of any countable collection of events is an event

e.g. Toss a coin twice

S (sample space) = {HH, HT, TH, TT}

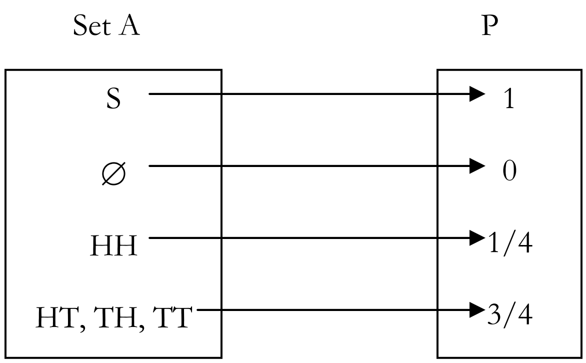
A = {S, $\varnothing$, HH, (HT, TH, TT})}
B = {S, $\varnothing$, (HT, TH)}

Is set A a σ-field? Is set B a σ-field?

The probability measure P must satisfy the following properties:

(a) If E is an event, then $0 \le P(E) \le 1$
(b) $P(S) = 1$
(c) If $E_1, E_2, \ldots$ is a countable collection of pair-wise disjoint events, then $P(\cup E_i) = \sum P(E_i)$

e.g.



Theorem

(1) If E is an event $\Rightarrow P(E^C) = 1 - P(E)$
(2) If A and B are events and $A \subset B \Rightarrow P(A) \le P(B)$
(3) If A and B are events and $A \subset B \Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$

3. Finite Sample Spaces

$P(A) = \dfrac{\# A}{\# S}$ (the hash "#" symbol means "to count"; #A (S): count elements in set A (sample space)).

By now, you should know why we need to learn some counting rules earlier.
e.g. Throw a fair dice, what is the probability that the number on the top face is no greater than four?

e.g. A normal deck of 52 cards is shuffled and one hand of five cards is dealt to Arlen.
(a) What is the probability that Arlen's hand is four of a kind?
(b) What is the probability that this hand is a full house?

e.g. The famous birthday problem: In a class of 30 students, what is the probability that at least two students share a common birthday?

4. Conditional Probability, Law of Total Probability, and Bayes' Theorem

(4.1) Conditional Probability

$P(A|B) = \dfrac{P(A \cap B)}{P(B)} \Leftrightarrow P(A\cap B) = P(A|B)*P(B) = P(B|A)*P(A)$

What does the above conditional probability mean exactly?

(4.2) Law of Total Probability

If $A_1, A_2, \ldots, A_n$ are collectively exhaustive and mutually exclusive, then

$$P(B) = \sum_{i=1}^{n} P(B \,|\, A_i) * P(A_i)$$

Collectively exhaustive $\Rightarrow \cup A_i = S$

Mutually exclusive $\Rightarrow A_i \cap A_j = \phi$

e.g. Throw a fair dice, what is the probability to observe a six given that the outcome is an even number?

e.g. Three bags:

Bag1: 1W, 3R, Bag 2: 2W, 2R, Bag 3: 4W

(a) Pick a bag at random and then select a ball at random, what is the probability to choose a red ball from bag 2?

(b) Pick a bag at random then select two balls without replacement, what is the probability that both are red?

Answer:

(a) Denote bag 2 (B2) as event A and red ball (R) as event B, then the question asks to find what's the probability of $P(A\cap B)$?

We can use formula: $P(A\cap B) = P(A|B)*P(B) = P(B|A)*P(A)$, but which one?

Substitute B2 and R for A and B respectively;

$P(B2\cap R) = P(B2|R)*P(R) = P(R|B2)*P(B2)$, so which one do you choose?

(b) Let B denote the event that there are 2 red balls (2R) and A_i denote the bag i (Bagi) is chosen.

$$P(2R) = \sum_{i=1}^{3} P(2R \,|\, Bagi) * P(Bagi)$$

Let's try one more example
e.g. There are four coins in a drawer. Two of them are fair, one has probability of 1/3 coming up heads, and the remaining one has the probability 3/4 coming up heads. Pick a coin at random and flip, what's the probability it comes up heads?

Independence
Def: A and B are independent $\Leftrightarrow$ $P(A \cap B) = P(A) * P(B)$

e.g. A fair coin is tossed three times. A: a head occurs on each of the first two tosses. B: a tail occurs on the third toss. C: exactly two tails occur in the three tosses.
(a) Are event A and B independent?
(b) Are event B and C independent?

(4.3) Bayes' Theorem
If $A_1, A_2, \ldots, A_n$ are collectively exhaustive and mutually exclusive, then

$$P(A_i | B) = \frac{P(B | A_i) * P(A_i)}{\sum_{j=1}^{n} P(B | A_j) * P(A_j)}$$

Let consider a simple case when j =2

$$P(A | B) = \frac{P(B | A) * P(A)}{P(B | A) * P(A) + P(B | A^C) * P(A^C)}$$

Does the denominator of the above formula look familiar to you?

e.g. Disease Testing

Let D = the subject has disease, T = the test reads positive, and P(D) = 0.5%

Q: We have a test which is 99% sensitive and 98% specific. If we apply the test to an individual and get a positive reading, what's the probability that he has the disease?

Definition:

$$\text{Sensitivity} = P(T | D)$$
$$\text{Specificity} = P(T^C | D^C)$$

What does this question ask? Can we show it in symbols & in a Venn diagram?
Term: PPV (Positive predictive value)

By the same token, what is the negative predictive value (NPV)? Do your calculation by hand.