Handout #2

Title: FAE | Spring/2015
Course: Econ 368/01 | Instructor: Dr. I-Ming Chiu

Introduction to Matrix: Matrix operations & Geometric meaning

Matrix: a rectangular array of numbers enclosed in parentheses (or square brackets). It is conventionally denoted by a capital letter. Matrix is a powerful tool to organize data. A lot of statistical methods involve the manipulation (i.e., transformation) of data matrix.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, the element in matrix A is denoted by $a_{i,j}$; the subscript i (i^{th} row) and j (j^{th} column) are indices that tell the location of element $a_{i,j}$. The largest number of i and j tells the dimension (order) of a matrix.

For example, $a_{1,2} = 2$ and matrix A is of order 2x2 (read as two by two)

Consider another matrix B, where the element is denoted by $b_{i,j}$. Since the largest i = 3 and largest j = 4, the matrix B is of order 3x4.

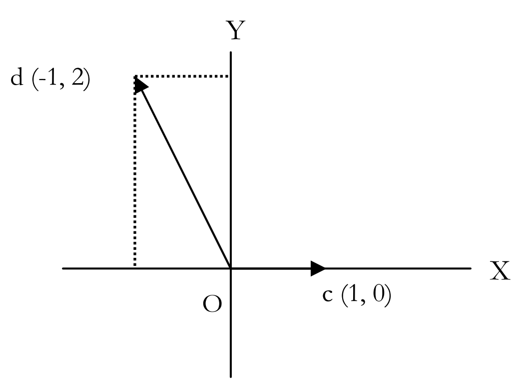
$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & -2 & 0 \\ 2 & 1 & 1 & -1 \end{bmatrix}$$

Q: what is the element of $b_{2,3}$ and $b_{3,2}$?

$c = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $d = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $e = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ (symbol $\vec{x}$, with an arrow, is often used in math class)

Matrix c is of order 1x2, d is of order 2x1 and e is of order 1x3. When there is only one row (column) in a matrix, it's termed as row (column) vector.

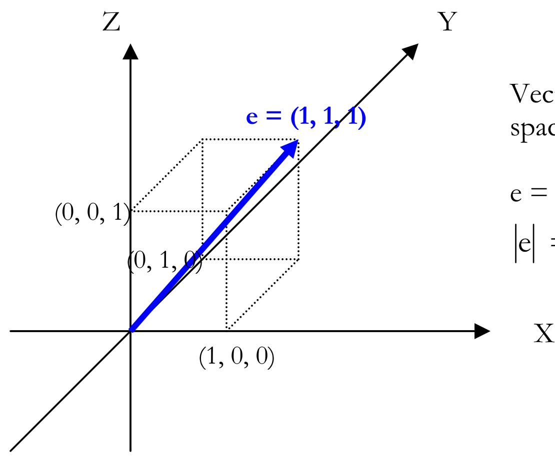
We can show the vector c & d in the X-Y plane (for e, three dimensional space is needed)



Suppose $c = [c_1, c_2, \ldots, c_n]$, the length (norm) of c is

$$|\vec{c}| = \sqrt{c_1^2 + c_2^2 + \ldots + c_n^2}$$

Q: Please find $|\vec{d}|$ in the left graph.



Vector e in a three-dimensional space (X, Y, Z).

e = (1, 1, 1) and its norm
$|e| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Algebra of Matrices

a) Equality of matrices

If A = B, then $a_{i,j} = b_{i,j}$. Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

b) Scalar multiplication

$k*A = \{k * a_{i,j}\}$. Example: $k = 3$, $k*A = 3*\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

c) Addition and Subtraction

$A \pm B = \{a_{i,j} + b_{i,j}\}$. Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 4 \end{bmatrix}$

d) Matrix multiplication

A (m x n) and B (n x p) must be conformable.

$$A*B = \begin{bmatrix} a_{11} & . & . & a_{1n} \\ a_{21} & . & . & a_{2n} \\ . & . & . & . \\ a_{m1} & . & . & a_{mn} \end{bmatrix} * \begin{bmatrix} b_{11} & . & . & b_{1p} \\ b_{21} & . & . & b_{2p} \\ . & . & . & . \\ b_{n1} & . & . & b_{np} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} * \sum_{j=1}^{n} b_{j1} & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \sum_{j=1}^{n} a_{mj} * \sum_{j=1}^{n} b_{jp} & . & . & . \end{bmatrix}$$

e.g.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} (1*0)+(2*-1) & (1*2)+(2*0) \\ (3*0)+(4*-1) & (3*2)+(4*0) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -4 & 6 \end{bmatrix}$$

e.g.

	Number of cars	Number of buses
Monday	30	5
Tuesday	25	5
Wednesday	35	15

Price = $4/car, $8/bus, find the revenue (R) on Monday, Tuesday, and Wednesday

$$R = \begin{bmatrix} 30 & 5 \\ 25 & 5 \\ 35 & 15 \end{bmatrix} * \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 160 \\ 140 \\ 260 \end{bmatrix}$$

Additional example

$$q = \begin{bmatrix} 15000 \\ 27000 \\ 13000 \end{bmatrix}, P = \begin{bmatrix} 10 & 12 & 5 \end{bmatrix}, z = \begin{bmatrix} 11000 \\ 30000 \end{bmatrix}, w = \begin{bmatrix} 20 & 8 \end{bmatrix}$$

Q: Please show the amount of profit in terms of matrix operation.

e.g.
$3x_1 + 2x_2 = 0$
$x_1 - x_2 = -2$

$$Ax = b, \text{ where } A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

A*B ≠ B*A, unless A = B.

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$. We can get the product of A*B but not B*A.

e) Rank of matrix: maximum number of independent rows or columns.

f) Transpose of a matrix (symbol: A^T or A')

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$, $B^T = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

Rules:
1) (A')' = A
2) (A ± B)' = A' ± B'
3) (A*B)' = B'*A'

g) Special matrices

g1: if the number of rows equals the number of columns, it is a **square** matrix.
g2: A is a square matrix, and if A = A', then A is a **symmetric** matrix

e.g., $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

Notice: For a symmetric matrix A, $a_{i,j} = a_{j,i}$ ($i \neq j$)

g3: **Diagonal** matrix (all the off-diagonal elements are zero)

e.g., $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

g4: **Identity** matrix (all the diagonal elements are one and off-diagonal elements are zero)

e.g., $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

g5: A square matrix A is **idempotent** if

$A = A^2 = A^3 = \dots$

e.g., $A = \begin{bmatrix} \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \\ \frac{-1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix}$, $A*A = \begin{bmatrix} \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \\ \frac{-1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix}$, A*A*A = still A (do not change)

h) <u>The trace of a square matrix A (nxn)</u>

trace (A) = a_{11} + …+ a_{nn} (the sum of all the diagonal elements)

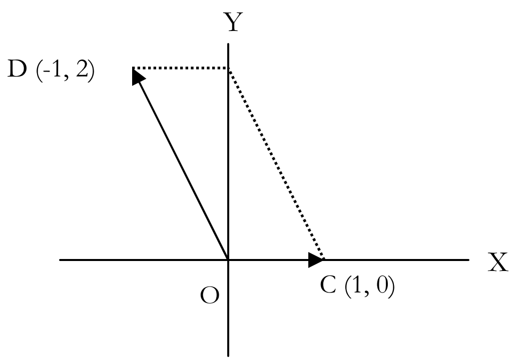
e.g., the trace of the previous idempotent matrix is one (= 1/6 + 2/3 + 1/6)

Q: What is the trace of a 3x3 identity matrix?

i) <u>Determinant of a square matrix</u>

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

det (A) or $|A|$ = a_{11}*a_{22} – a_{12}*a_{21} (show geometric meaning)



Suppose $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

det (A) = 2 – 0 = 2

Suppose A is a nxn matrix, where n ≥ 3, how do we find det (A)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(Cofactor) Expansion approach:
Step 1: choose any column or row
Step 2: find the minor (determinant of a sub-matrix) of each element
Step 3: find the cofactor
Step 4: multiply each element by its cofactor and get the sum of these products

Step 1: choose the first row $\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$

Step 2: Minor: A_{11} = det ($\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{bmatrix}$) = det ($\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$)

$A_{12} = \det \left(\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{bmatrix} \right) = \det \left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \right)$

Step 3: Cofactor $C_{i,j} = (-1)^{i+j} * A_{i,j}$

$C_{12} = (-1)^{1+2} * A_{12}$

Step 4: $\det(A) = \sum_{j=1}^{3} a_{i,j} * C_{i,j}$ (or $\sum_{i=1}^{3} a_{i,j} * C_{i,j}$ if expanded by column)

e.g., $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$

$A_{11} = \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\right) = 5$, $C_{11} = (-1)^{1+1} * A_{11} = 5$

$A_{12} = \det\left(\begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}\right) = -3$, $C_{12} = (-1)^{1+2} * A_{12} = 3$

$A_{13} = \det\left(\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}\right) = -1$, $C_{13} = (-1)^{1+3} * A_{13} = -1$

$\det(A) = (1*5) + (-1*3) + (0*-1) = 2$

j) <u>Inversion of a square matrix (A^{-1})</u>

Definition: $A^{-1} * A = I$

e.g., $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, let $A^{-1} = \begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix}$

$A^{-1} * A = \begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix} * \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, four equations and four unknowns; we need to solve a simultaneous equation system in order to find all the elements in A^{-1}

How do we find the inverse of a square matrix A (nxn) if $n \geq 2$?

Formula:

$A^{-1} = \frac{1}{\det(A)}$ *adj (A), where adj (A) is termed adjoint matrix

$$\text{adj }(A) = (\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix})^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Cofactor $C_{i,j} = (-1)^{i+j} * A_{i,j}$

e.g.,

$$A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\det(A) = -5, \text{ adj }(A) = \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} * \text{adj }(A) = -\frac{1}{5} * \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{5} \end{bmatrix}$$

Double check:

$$A * A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} * \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Q: find the inverse of the following 3x3 matrix B (B^{-1}?)

$$B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Rules:

1) $A^{-1} * A = A * A^{-1} = I$
2) $(A^{-1})^{-1} = A$
3) $(A*B)^{-1} = B^{-1} * A^{-1}$
4) $(A')^{-1} = (A^{-1})'$
5) $\det(A^{-1}) = \frac{1}{\det(A)}$

Solve Simultaneous Linear Equations

$a_{11}*x_1 + a_{12}*x_2 = b_1$
$a_{21}*x_1 + a_{22}*x_2 = b_2$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$\Rightarrow A*x = b$

$\Rightarrow A^{-1}*A*x = A^{-1}*b$

$\Rightarrow I*x = A^{-1}*b$

$\Rightarrow x = A^{-1}*b$ (we solve x using inversion approach)
Earlier example:
$3x_1 + 2x_2 = 0$
$x_1 - x_2 = -2$

$$A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$x^* = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{5} \end{bmatrix} * \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \frac{6}{5} \end{bmatrix}$$

e.g. Supply and demand model

$Q = 10 - P$
$Q = 2 + P$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, x = \begin{bmatrix} P \\ Q \end{bmatrix}, b = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

$$x^* = \begin{bmatrix} P^* \\ Q^* \end{bmatrix} = A^{-1}*b = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} * \begin{bmatrix} 10 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

k) <u>Eigenvalue Problem</u>

Eigenvalue is also called <u>latent value</u> or <u>characteristic root</u>.

$$Aq = \lambda q$$

A is a known nxn **<u>symmetric</u>** matrix, λ is an unknown scalar and q is an unknown (nx1) column vector. The solution of finding the unknown scalar λ (Eigenvalue) and the unknown q (Eigenvector) is called the Eigenvalue problem.

e.g. $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$

$(A - \lambda I)*q = 0$

For a nontrivial solution q, "$A - \lambda I$" (a 2x2 matrix) must be singular. It means that
$\det (A - \lambda I) = 0$

$A - \lambda I = \begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix}$

$\det (A - \lambda I) = 0 \Rightarrow (2 - \lambda)(-1 - \lambda) - 4 = 0 \Rightarrow$ solve for λ and obtain $\lambda_1 = 3$ and $\lambda_2 = -2$

The **number** of **non-zero** eigenvalues can be used to find the rank of the matrix. In this example, there are two non-zero eigenvalues. Therefore, the matrix A has full rank (i.e., rank (A) = 2) or it means there are two linear independent rows (columns) in matrix A.

Next we have to find the eigenvector:

When $\lambda_1 = 3$

$(A - \lambda_1 I)q_1 = 0 \Rightarrow -q_{11} + 2q_{21} = 0 \Rightarrow q_{21} = (1/2)\, q_{11}$, where $q_1 = [q_{11}, q_{21}]^T$

Use the normalization condition: $q_{11}^2 + q_{21}^2 = 1$

Solve for q_{11} and q_{21} we can obtain: $q_{11} = \frac{2}{\sqrt{5}}$ $q_{21} = \frac{1}{\sqrt{5}}$

Eigenvector $q_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ … corresponding to $\lambda_1 = 3$

When $\lambda_2 = -2$

$(A - \lambda_2 I)q_2 = 0 \Rightarrow 4q_{12} + 2q_{22} = 0 \Rightarrow q_{22} = (-2)q_{12}$, where $q_2 = [q_{12}, q_{22}]^T$

Use the normalization condition: $q_{12}^2 + q_{22}^2 = 1$

Solve for q_{12} and q_{22}: $q_{12} = \frac{1}{\sqrt{5}}$, $q_{22} = \frac{-2}{\sqrt{5}}$

Eigenvector $q_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix}$… corresponding to $\lambda_1 = -2$

Property of q_1 and q_2: $q_1^T q_1 = 1$, $q_1^T q_2 = 0$

Collect q_1 and q_2 in a matrix Q. Where $Q = [q_1, q_2]$

Properties of Q:

a) $Q^TQ = QQ^T = I$ (from previous property), $Q^T = Q^{-1} \Leftrightarrow$ Q is an **Orthogonal matrix**

b) $Q^TAQ = \Lambda$, where $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Property (b) above is called "**Diagonalization**".

c) If A is not symmetric then $Q^{-1}AQ = \Lambda$. Q is no longer Orthogonal.

d) trace (A) = trace (Λ)

e) det (A) = $\prod \lambda_i = \lambda_1 * \lambda_2 \ldots * \lambda_n$

Connect the above results and we have the following conclusions:

If an nxn matrix A is nonsingular $\Leftrightarrow$ iff
- a) det (A) $\neq 0$
- b) A^{-1} exists
- c) rank (A) = n (full rank)

Note: "iff": if and only if; an equivalent (necessary and sufficient) statement symbol.