Handout #4

Title: Foundations of Econometrics Fall/2015
Course: Econ 367 Instructor: Dr. I-Ming Chiu

Imagine that we can transform the data into a form where we can operate on and therefore enabling us to analyze it. Matrix and matrix operations help achieve the above goal once we design the algorithms according to the analytical objectives. Let's use X to denote our data set where the observations (cases) are in rows and variables are in columns.

$$X\,[X_{i,j}] = \begin{bmatrix} X_{1,1} & . & . & X_{1,p} \\ X_{2,1} & . & . & X_{2,p} \\ . & . & . & . \\ X_{n,1} & . & . & X_{n,p} \end{bmatrix}; \text{ where } i = 1\ldots n,\ j = 1\ldots p.$$

All of the elements in X are in numerical mode. We can always recode the categorical variables into numerical mode, e.g., female = 1 and male = 0 or Yes = 1 and No = 0, etc.

Introduction to Matrix

Matrix: a rectangular array of numbers enclosed in parentheses (or square brackets). It is conventionally denoted by a capital letter. Matrix is a powerful tool to organize data. A lot of statistical methods involve the manipulation (i.e., transformation) of data matrix.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, the element in matrix A is denoted by $a_{i,j}$; the subscript i (i^{th} row) and j (j^{th} column) are indices that tell the location of element $a_{i,j}$. The largest number of i and j tells the dimension (order) of a matrix.

For example, $a_{1,2} = 2$ and matrix A is of order 2x2 (read as two by two)

Consider another matrix B, where the element is denoted by $b_{i,j}$. Since the largest i = 3 and largest j = 4, the matrix B is of order 3x4.

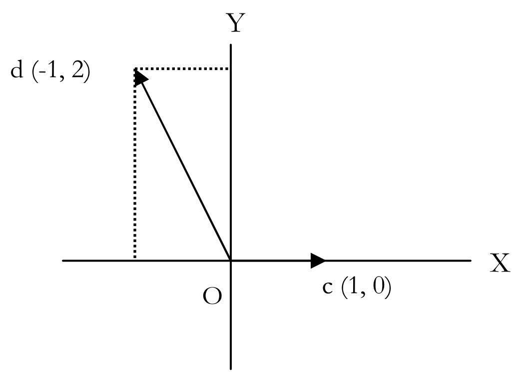
$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & -2 & 0 \\ 2 & 1 & 1 & -1 \end{bmatrix}$$

Q: what is the element of $b_{2,3}$ and $b_{3,2}$?

$c = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $d = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $e = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ (symbol $\vec{c}$, with an arrow, is often used in math class)

Matrix c is of order 1x2, d is of order 2x1 and e is of order 1x3. When there is only one row (column) in a matrix, it's termed as a row (column) vector.

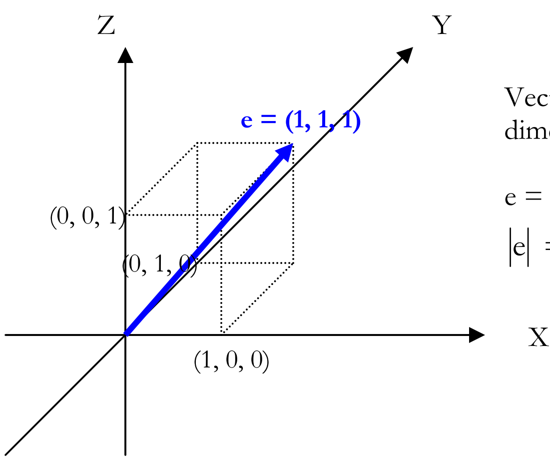
We can show the vector c & d in the X-Y plane (for e, three dimensional space is needed)



Suppose c = [c_1, c_2, … , c_n], the length (norm) of c is

$$|\vec{c}| = \sqrt{c_1^{\,2} + c_2^{\,2} + \ldots + c_n^{\,2}}$$

Q: Please find $|\vec{d}|$ in the left graph.



Vector e exists in a three-dimensional space (X, Y, Z).

e = (1, 1, 1) and its norm is

$$|e| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Algebra of Matrices

a) Equality of matrices

If A = B, then $a_{i,j} = b_{i,j}$. Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

b) Scalar multiplication

k*A = $\{k * a_{i,j}\}$. Example: k = 3, k*A = 3*$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

c) Addition and Subtraction

A ± B = $\{a_{i,j} + b_{i,j}\}$. Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 4 \end{bmatrix}$

d) Matrix multiplication

A (m x n) and B (n x p) must be conformable.

$$A*B = \begin{bmatrix} a_{11} & . & . & a_{1n} \\ a_{21} & . & . & a_{2n} \\ . & . & . & . \\ a_{m1} & . & . & a_{mn} \end{bmatrix} * \begin{bmatrix} b_{11} & . & . & b_{1p} \\ b_{21} & . & . & b_{2p} \\ . & . & . & . \\ b_{n1} & . & . & b_{np} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} * \sum_{j=1}^{n} b_{j1} & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \sum_{j=1}^{n} a_{mj} * \sum_{j=1}^{n} b_{jp} & . & . & . \end{bmatrix}$$

e.g.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} (1*0)+(2*-1) & (1*2)+(2*0) \\ (3*0)+(4*-1) & (3*2)+(4*0) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -4 & 6 \end{bmatrix}$$

e.g.

	Number of cars	Number of buses
Monday	30	5
Tuesday	25	5
Wednesday	35	15

Price = \$4/car, \$8/bus, find the revenue (R) on Monday, Tuesday, and Wednesday

$$R = \begin{bmatrix} 30 & 5 \\ 25 & 5 \\ 35 & 15 \end{bmatrix} * \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 160 \\ 140 \\ 260 \end{bmatrix}$$

Additional example

$$q = \begin{bmatrix} 15000 \\ 27000 \\ 13000 \end{bmatrix}, P = \begin{bmatrix} 10 & 12 & 5 \end{bmatrix}, z = \begin{bmatrix} 11000 \\ 30000 \end{bmatrix}, w = \begin{bmatrix} 20 & 8 \end{bmatrix}$$

Q: Please show the amount of profit in terms of matrix operation.

e.g.
$3x_1 + 2x_2 = 0$
$x_1 - x_2 = -2$

$$Ax = b, \text{ where } A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$A*B \neq B*A$, unless A = B.

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$. We can get the product of A*B but not B*A.

e) Rank of matrix: maximum number of independent rows or columns.

f) Transpose of a matrix (symbol: A^T or A')

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}, B^T = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

Rules:
1) (A')' = A
2) (A ± B)' = A' ± B'
3) (A*B)' = B'*A'

g) Special matrices

g1: if the number of rows equals the number of columns, it is a **square** matrix.
g2: A is a square matrix, and if A = A', then A is a **symmetric** matrix

$$\text{e.g., } A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Notice: For a symmetric matrix A, $a_{i,j} = a_{j,i}$ $(i \neq j)$

g3: **Diagonal** matrix (all the off-diagonal elements are zero)

e.g., $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

g4: **Identity** matrix (all the diagonal elements are one and off-diagonal elements are zero)

e.g., $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

g5: A square matrix A is **idempotent** if

$A = A^2 = A^3 = \ldots$

e.g., $A = \begin{bmatrix} \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \\ \frac{-1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix}$, $A*A = \begin{bmatrix} \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \\ \frac{-1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix}$, A*A*A = still A (do not change)

h) <u>The trace of a square matrix A (nxn)</u>

trace (A) = $a_{11} + \ldots + a_{nn}$ (the sum of all the diagonal elements)

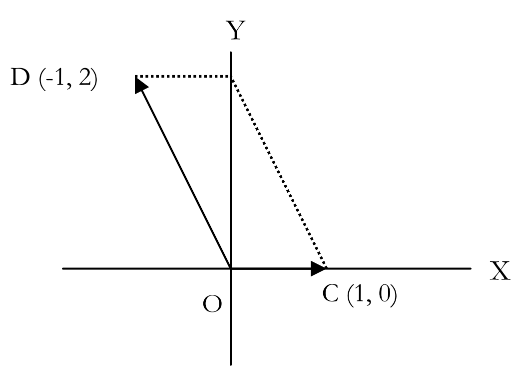
e.g., the trace of the previous idempotent matrix is one (= 1/6 + 2/3 + 1/6)

Q: What is the trace of a 3x3 identity matrix?

i) <u>Determinant of a square matrix</u>

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

det (A) or $|A| = a_{11}*a_{22} - a_{12}*a_{21}$ (show geometric meaning)



Suppose $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

det (A) = 2 – 0 = 2

Suppose A is a nxn matrix, where n ≥ 3, how do we find det (A)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(Cofactor) Expansion approach:
Step 1: choose any column or row
Step 2: find the minor (determinant of a sub-matrix) of each element
Step 3: find the cofactor
Step 4: multiply each element by its cofactor and get the sum of these products

Step 1: choose the first row $\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$

Step 2: Minor: A_{11} = det ($\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$) = det ($\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$)

A_{12} = det ($\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$) = det ($\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$)

Step 3: Cofactor $C_{i,j} = (-1)^{i+j} * A_{i,j}$

$C_{12} = (-1)^{1+2} * A_{12}$

Step 4: det (A) = $\sum_{j=1}^{3} a_{i,j} * C_{i,j}$ (or $\sum_{i=1}^{3} a_{i,j} * C_{i,j}$ if expanded by column)

e.g., $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$

A_{11} = det ($\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$) = 5, $C_{11} = (-1)^{1+1} * A_{11} = 5$

A_{12} = det ($\begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$) = -3, $C_{12} = (-1)^{1+2} * A_{12} = 3$

A_{13} = det ($\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$) = -1, $C_{13} = (-1)^{1+3} * A_{13} = -1$

det (A) = (1*5) + (-1*3) + (0*-1) = 2

j) Inversion of a square matrix (A^{-1})

Definition: $A^{-1}*A = I$

e.g., $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, let $A^{-1} = \begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix}$

$A^{-1}*A = \begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix} * \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, four equations and four unknowns; we need to solve a simultaneous equation system in order to find all the elements in A^{-1}

How do we find the inverse of a square matrix A (nxn) if $n \geq 2$?

Formula:

$A^{-1} = \frac{1}{\det(A)} *\text{adj}(A)$, where adj (A) is termed adjoint matrix

$$\text{adj}(A) = (\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix})^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Cofactor $C_{i,j} = (-1)^{i+j} * A_{i,j}$

e.g.,

$A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$

det (A) = -5, adj (A) = $\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$

$$A^{-1} = \frac{1}{\det(A)} *\text{adj}(A) = -\frac{1}{5} * \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{5} \end{bmatrix}$$

Double check:

$$A*A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} * \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Q: find the inverse of the following 3x3 matrix B (B^{-1}?)

$$B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Rules:
1) $A^{-1}*A = A*A^{-1} = I$
2) $(A^{-1})^{-1} = A$
3) $(A*B)^{-1} = B^{-1}*A^{-1}$
4) $(A')^{-1} = (A^{-1})'$
5) $\det(A^{-1}) = \frac{1}{\det(A)}$

Solve Simultaneous Linear Equations

$a_{11}*x_1 + a_{12}*x_2 = b_1$
$a_{21}*x_1 + a_{22}*x_2 = b_2$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$\Rightarrow A*x = b$

$\Rightarrow A^{-1}*A*x = A^{-1}*b$

$\Rightarrow I*x = A^{-1}*b$

$\Rightarrow x = A^{-1}*b$ (we solve x using inversion approach)
Earlier example:
$3x_1 + 2x_2 = 0$
$x_1 - x_2 = -2$

$$A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$x^* = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{5} \end{bmatrix} * \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \frac{6}{5} \end{bmatrix}$$

e.g. Supply and demand model

Q = 10 – P
Q = 2 + P

A = $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, x = $\begin{bmatrix} P \\ Q \end{bmatrix}$, b = $\begin{bmatrix} 10 \\ 2 \end{bmatrix}$

$$x^* = \begin{bmatrix} P^* \\ Q^* \end{bmatrix} = A^{-1}*b = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} * \begin{bmatrix} 10 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

k) <u>Eigenvalue Problem</u>

We'll learn eigenvalue when it's needed.

l) Matrix Decomposition Methods
A short supplement will be written for your reference in early December.

Note: we will use the "Mata" module in the STATA for conducting matrix operations.