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ORIGINAL PAPER

Convergence of Markovian price processes in a financial market transaction model

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Abstract This paper studies a financial market transaction model and convergence of Markovian price processes generated by an α-double auction in Xu et al. (Expert Syst Appl 41(16):7032–7045, 2014) and extends their results for a fixed α in [0, 1] to the case where α is governed by a time non-homogeneous Markov chain over a set of finite states defined by $R \equiv \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_r \le 1$. A convergence result similar to that in Xu et al. (2014) holds, with the fixed α replaced with the average $\alpha^* = \frac{1}{r}\sum_{\theta=1}^{r} \alpha_\theta$. We also identify the conditions under which a price process generated by such a Markovian α-double auction converges in probability to a Walrasian equilibrium of the underlying financial market transaction model. A number of simulations are conducted and these simulations are consistent with the theoretical results of the paper.

Keywords Double auctions · Bubble and crash · Incremental subgradient methods · Sentiments · Walrasian equilibrium

This paper supersedes the paper entitled "Markovian α-Double Auctions".

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1 Introduction

In a large class of quasilinear economies, price adjustment processes generated by the subgradient method (under the Walrasian hypothesis via Fenchel's duality theorem) (e.g., Bertsekas and Tsitsiklis 2000) converge to a Walrasian equilibrium when the step size is diminishing (see footnote 2 for definition). Because the adjustment in prices under the Walrasian hypothesis depends on the total demand and supply of the economy, such a price process is a centralized system that has a limited use for a large scale economy where information on the total demand and supply is costly obtained. In contrast, the incremental subgradient method, originally proposed in Kibardin (1980), is a decentralized system that uses only the information on individual's demand and supply. For example, in a cyclic incremental subgradient method (e.g., Nedić and Bertsekas 2001, and see Bertsekas 2010 for a survey), agents form a ring or cyclic structure and prices are iterated along the ring, one agent at a time. That is, only one agent's demand and supply (at the prices that are newly updated along the ring) feeds into the price process at each iteration. It is not necessary to have the same ring structure for a price process but the ring structure must stay put until the price process has incorporated all individual's demand and supply along the same ring. Such a decentralized system is useful to study the convergence of price processes for a large scale economy because there is no need to obtain information about the total demand and supply. More importantly, under the same diminishing step size rule, a price process generated under the cyclic incremental subgradient method always converges to a Walrasian equilibrium of the underlying economy, as shown in Nedić and Bertsekas (2001). Such a convergence result in Nedić and Bertsekas (2001) can be shown for the randomized incremental subgradient method, where each agent is randomly drawn, with equal probability. It can also be extended to the case when all agents form a Markovian network specified in Ram et al. (2009) (see Assumption 3.1 in Sect. 3 for detail).

This paper studies a financial market transaction model given by a general form below (Bertsekas 2010):

$$\mathcal{P} \qquad \text{minimize}_{y\in Y}\ F(y) \equiv \sum_{i=1}^{m} f_i(y) + \sum_{j=1}^{n} g_j(y).$$

For all sellers $i \in I = \{1, 2, \ldots, m\}$ and buyers $j \in J = \{1, 2, \ldots, n\}$, f_i: $R^d \to R$ and g_j: $R^d \to R$ are real-valued (possibly non-differentiable) convex functions and Y is a nonempty convex subset of R^d. For a large class of quasilinear economies for selling a single divisible good, $f_i(y)$ is the producer's surplus for seller i at price y and $g_j(y)$ is the consumer's surplus for buyer j at price y; the gradient $\bigtriangledown f_i(y)$ is the quantity supplied for seller i at price y and the gradient $-\bigtriangledown g_j(y)$ is the quantity demanded for buyer j at price y; and a price y is Walrasian if and only if y is a solution to problem $\mathcal{P}$ (Ma and Nie 2003). These observations can be extended to an exchange economy for (multiple) indivisible goods in Bikhchandani and Mamer (1997), typified by the well-known job matching model of Kelso and Crawford (1982), when the duality gap is zero. With the gross substitutes preferences in Kelso and

Crawford (1982), a solution y to problem $\mathcal{P}$ is also a Walrasian equilibrium for such an exchange economy with (multiple) indivisible goods. For an exchange economy in Bikhchandani and Mamer (1997), the problem $\mathcal{P}$ is exactly the dual problem of a linear programming that finds an efficient allocation; and Bikhchandani and Mamer (1997) show that a Walrasian equilibrium exists if and only if the duality gap is zero (also see Ma and Nie 2003; Ma and Li 2011).

We are interested in the convergence of the price processes generated by a double auction in Ma and Li (2011) in an economy that has a dual $\mathcal{P}$ specified above. The two double auctions in Ma and Li (2011) specify two different ways how the two sequences of bid and ask orders enter a price process, largely motivated by the literature of the incremental subgradient methods, e.g., Kibardin (1980), Nedić and Bertsekas 2001), and Ram et al. (2009). A double auction in Ma and Li (2011) has a two ring structure, one ring for the sellers (generating a sequence of asks) and one ring for the buyers (generating a sequence of bids). At each iteration, the sequence of asks is updated by incorporating individual seller's supply information along the sellers' ring and the sequence of bids is updated by incorporating individual buyer's demand information along the buyers' ring, one pair at each moment of time. The price process is formed as a weighted average of a pair of ask and bid, with a weight $\alpha \in [0, 1]$, similar to a double auction in a real exchange market. An important feature of such a double auction, different from that in the cyclic incremental subgradient method, is that sellers and buyers can have different step sizes or step size rules. This feature has been captured by a parameter λ in a condition related to the two different step sizes (see the inequality (7.2) about the λ below). The weight α and the parameter λ play an important role for the convergence of price processes generated by a double auction (Ma and Li 2011; Xu et al. 2014). Our main convergence result in Theorem 7.1 reaches the same conclusion.

The rest paper is organized as follows. Section 2 introduces Markovian price processes nontechnically. Section 3 defines the Markovian chain and a key assumption on the transition probability matrices. Section 4 introduces the model and an example how the model is related to an exchange economy. Section 5 defines a Markovian α-double auction formally. Section 6 provides a simple example and reports some simulation results. Section 7 proves the main theorem, which covers two cases: one with noises and the other without noises. Section 8 provides some remarks and additional experiments. Section 9 concludes.

2 Markovian price processes

We now introduce in more detail the Markovian price processes studied in this paper. Some technical details are introduced in Sects. 3–5. In a stock exchange market, a bid (an ask) order for a stock at each moment consists of a pair of a bid and a bid size (an ask and an ask size). A bid order indicates the buyer, by submitting his bid order, is willing to buy a number of shares no greater than the quantity of the bid size at a unit share price no greater than the bid; and an ask order indicates the seller, by submitting her ask order, is willing to sell a number of shares no greater than the ask size at a unit share price no less than the ask. Therefore, a

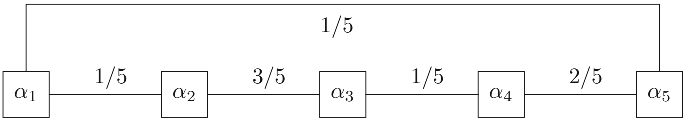

Fig. 1 An example of Markovian α and transition probabilities

trade is only possible at each moment of time between a bid order and an ask order and only executed if the bid is at least as high as the ask. Because the bid is no less than the ask of an executed trade, the share price of a trade is often settled in the interval of [ask, bid]. The price rule implemented in such a manner in a clearinghouse is called an α-double auction. That is, the price that is settled equals a weighted average of the bid and the ask for some weight $\alpha \in [0,1]$. Specifically, if we use x_{k+1}, ψ_{k+1} and φ_{k+1} to denote the executed price, the ask and the bid at time $k+1$, respectively, with the bid no less than the ask, then the price x_{k+1} of the executed trade is determined by

$$x_{k+1} = \alpha_{k+1}\psi_{k+1} + (1-\alpha_{k+1})\varphi_{k+1}, \quad \alpha_{k+1} \in [0,1]. \tag{2.1}$$

Such an α-double auction[1] has been studied in Chatterjee and Samuelson (1983), Myerson and Satterthwaite 1983), and Wilson (1985) in theory for selling a single unit of an indivisible object for a fixed $\alpha \in [0,1]$ in a strategic form game with incomplete information. In practice, however, an α-double auction is dynamic, with a time-varying α, and the number of shares in each trade often involves more than a single unit.

Figure 1 provides an illustration how α may evolve with time with five different states of αs, $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_5 \le 1$. The transition probability matrix $P_R(k)$ over the states in Fig. 1 at time k may be given by

$$P_R(k) = \begin{pmatrix} 3/5 & 1/5 & 0 & 0 & 1/5 \\ 1/5 & 1/5 & 3/5 & 0 & 0 \\ 0 & 3/5 & 1/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 2/5 & 2/5 \\ 1/5 & 0 & 0 & 2/5 & 2/5 \end{pmatrix}.$$

As in Ram et al. (2009), these transition probability matrices are doubly stochastic and depend on time k, $k = 0, 1, 2, \ldots$, in our general convergence results. The networks in Fig. 1 may also evolve across time in the sense that some links among these αs may drop off or be added occasionally as the auction proceeds.

Next we specify how the ask ψ_{k+1} and the bid φ_{k+1} in (2.1) are associated with the ask and bid sizes. Let x_k be the price of a stock at time k, the bid size at time $k+1$ of the bid order is the quantity demanded at x_k for the buyer and the ask size of the ask order at time $k+1$ is the quantity supplied at x_k for the seller. We also allow the buyer and the seller to submit bid and ask sizes that are contaminated by noises,

[1] This is the same as the k-double auction but k has been reserved to denote iterations in this paper.

which capture the idea that the buyer and the seller may not know precisely their true demand and supply. That is, the buyer and the seller know their demand and supply, but with some uncertainty, which is reflected into their bid size and ask size with a noise term. The bid φ_{k+1} and the ask ψ_{k+1} follow the equations

$$\psi_{k+1} = x_k - a_k \cdot \text{ask size}, \quad \varphi_{k+1} = x_k + b_k \cdot \text{bid size}, \tag{2.2}$$

where a_k and b_k are the ask and bid step sizes at k, respectively. The ask step size a_k is the price decrement of one share for the seller and the bid step size b_k is the price increment for one share for the buyer. In a Markovian α-double auction of this study, these step sizes are time-varying and diminishing.[2] Note however, that bid and ask step sizes are not the same as the bid and ask spread $\varphi_{k+1} - \psi_{k+1}$.

Our major concern is that, under what conditions on $\{a_k\}$ and $\{b_k\}$, the Markovian price process defined in (2.1) and (2.2) converges. Moreover, we address the question as to how the convergence is related to the Walrasian equilibrium of the underlying exchange economy defined by $\mathcal{P}$, when αs evolve in a time non-homogeneous Markov chain, similar to the one shown in Fig. 1.

To answer the question, we have to consider the following alternative problem

$$\mathcal{P}(\alpha, \lambda) \qquad \text{minimize}_{y \in Y}\ F(y, \alpha, \lambda) \equiv \alpha \sum_{i=1}^{m} f_i(y) + \lambda(1-\alpha) \sum_{j=1}^{n} g_j(y),$$

where λ is a positive scalar. Let $Y^*(\alpha, \lambda)$ denote the set of solutions to $\mathcal{P}(\alpha, \lambda)$. Note that $Y^*(\alpha, \lambda)$ is also the set of solutions to $\mathcal{P}$ for $\alpha = \lambda(1-\alpha)$. For other values with inequality $\alpha \neq \lambda(1-\alpha)$, the two solution sets may differ. For a given α, Ma and Li (2011) identified two α-double auctions that generate a price process in (2.1) and (2.2) that converges to a solution in $Y^*(\alpha, \lambda)$ if the following holds

$$\sum_{k=0}^{\infty} \left| \frac{1}{n} b_k - \lambda \frac{1}{m} a_k \right| < +\infty$$

for some positive scalar λ. It is easier to understand what is this parameter λ by assuming that the limit of the step size ratio $\lim \frac{b_k}{a_k}$ exists. Then the unique λ must be equal to $\frac{m}{n} \lim \frac{b_k}{a_k}$; see Ma and Li (2011) for detail. Thus, this parameter λ reflects the aggressiveness in the way how buyers and sellers place their bid and ask orders (for one share) in the limit. The number n of buyers and the number m of sellers also matter. But λ is not related to f_i and g_j. Such a feature is important because f_i and g_j are privately known to sellers and buyers, respectively, not known to the mechanism designer. Nonetheless, without knowing any information about f_i and g_j, the two parameters α and λ can determine if the price process in (2.1) and (2.2) converges to a solution to $\mathcal{P}$.

Xu et al. (2014) extends the convergence results in Ma and Li (2011) to the case where sellers and buyers form two time non-homogeneous Markov chains, one for the sellers and the other for the buyers, in a style of forming local interactive

[2] (1) $a_k > 0$ and $b_k > 0$; (2) $\sum_{k=0}^{\infty} a_k = +\infty$ and $\sum_{k=0}^{\infty} b_k = +\infty$; (3) $\sum_{k=0}^{\infty} a_k^2 < +\infty$ and $\sum_{k=0}^{\infty} b_k^2 < +\infty$. See Nedić and Bertsekas (2001) and Ram et al. (2009).

Table 1 Experimental results for $m = 5$ and $n = 5, 10, 20$

	$m = 5$ $n = 5$		$m = 5$ $n = 10$				$m = 5$ $n = 20$			
Y^*	1.0000	1.0000	1.8028	1.8028	1.8028	1.8028	3.3912	3.3912	3.3912	3.3912
α^*	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2.2}{3}$	$\frac{2.2}{3}$
$Y^*(\alpha^*, 1)$	1.0000	1.0000	1.8028	1.8028	1.2748	1.2748	3.3912	3.3912	2.0449	2.0449
Noises	No	(0, 25)	No	(0, 25)	No	(0, 25)	No	(0, 25)	No	(0, 25)
Mean $\bar{\mu}$	1.0000	1.0012	1.8039	1.8101	1.2795	1.2732	3.4046	3.4286	2.0462	2.0572
SD $\bar{\sigma}$	0.0016	0.0018	0.0022	0.0015	0.0036	0.0023	0.0054	0.0052	0.0093	0.0065
$\bar{\mu} - 1.96\bar{\sigma}$	0.9968	0.9977	1.7995	1.8071	1.2726	1.2688	3.3940	3.4184	2.0281	2.0445
$\bar{\mu} + 1.96\bar{\sigma}$	1.0032	1.0047	1.8083	1.8131	1.2865	1.2776	3.4153	3.4389	2.0644	2.0700

$\lambda = 1$, $a_k = \frac{1}{k+1}$ and $b_k = \frac{\lambda n}{m} a_k$

Y^* is the equilibrium price of the original economy. The price process $\{x_k\}$ converges to $Y^*(\alpha^*, 1)$ with probability 1, according to Theorem 7.1. Weight in α^*, higher than $\frac{1}{2}$, results in a price that is lower than the equilibrium price Y^*. In each experiment, we have conducted 100 rounds, with x_0 generated uniformly from [1, 5], and each round has 20,000 iterations. The mean $\bar{\mu}$ and the standard deviation $\bar{\sigma}$ of $\{x_k\}$ for each experiment are derived from the average sample of 100 rounds for iterations from $k = 10{,}000$ to $k = 20{,}000$

networks in Ellison (1993) and Ram et al. (2009). Nevertheless, the weight α in Xu et al. (2014) remains a constant in [0, 1]. Because the clearinghouse in an exchange in practice often executes trades with various $\alpha \in [0, 1]$, most of the times, near the two end values 0 and 1, the current paper extends the results in Xu et al. (2014) to the case when α is time-varying. For a set of states defined by $R \equiv \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq 1$, define $\alpha^* = \frac{1}{r}\sum_{\theta=1}^{r} \alpha_\theta$. With the doubly stochastic transition probability matrix on R, as the one specified in Ram et al. (2009), we prove that a convergence result similar to those in Xu et al. (2014) holds, with α in Xu et al. (2014) replaced by this new α^*. Our results show that a Markovian α-double auction converges to the set of Walrasian equilibrium prices of the original economy if, in addition, the condition $\alpha^* = \lambda(1 - \alpha^*)$ holds. However, if the condition $\alpha^* = \lambda(1 - \alpha^*)$ does not hold, a Markovian α-double auction may converge to a price vector that is above or below a Walrasian equilibrium, which implies that there may exist a bubble (above the fundamental value) or a crash (below the fundamental value) for some goods. We provide simulations for a bubble and a crash with different α^* in Fig. 4a, b in Sect. 8. More experimental results about bubbles and crashes are founded in Tables 1 and 2 of Sect. 8.5.

3 Markov chains

We need to be precise about the way how bid and ask orders are cleared and enter the price of a stock in (2.1) and (2.2). We assume that sellers in I and buyers in J are connected in two time-varying networks, as in Xu et al. (2014), motivated largely by Ellison (1993) and Ram et al. (2009). Some links in the two networks may be lost or added over time occasionally. We follow Ram et al. (2009) to model these two networks

Table 2 Experimental results for $n = 5$ and $m = 5, 10, 20$

	$n=5$ $m=5$		$n=5$ $m=10$				$n=5$ $m=20$			
Y^*	1.0000	1.0000	0.5547	0.5547	0.5547	0.5547	0.2949	0.2949	0.2949	0.2949
α^*	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$
$Y^*(\alpha^*, 1)$	1.0000	1.0000	0.5547	0.5547	0.7845	0.7845	0.2949	0.2949	0.5898	0.5898
Noises	No	(0, 25)	No	(0, 25)	No	(0, 25)	No	(0, 25)	No	(0, 25)
Mean $\bar{\mu}$	1.0000	1.0012	0.5582	0.5587	0.7897	0.7913	0.3179	0.3167	0.6235	0.6371
SD $\bar{\sigma}$	0.0016	0.0018	0.0040	0.0026	0.0023	0.0025	0.0080	0.0065	0.0047	0.0177
$\bar{\mu} - 1.96\bar{\sigma}$	0.9968	0.9977	0.5504	0.5537	0.7851	0.7865	0.3021	0.3039	0.6144	0.6024
$\bar{\mu} + 1.96\bar{\sigma}$	1.0032	1.0047	0.5660	0.5637	0.7943	0.7962	0.3336	0.3295	0.6327	0.6717

$\lambda = 1$, $a_k = \frac{1}{k+1}$ and $b_k = \frac{\lambda n}{m} a_k$

Y^* is the equilibrium price of the original economy. The price process $\{x_k\}$ converges to $Y^*(\alpha^*, 1)$ with probability 1, according to Theorem 7.1. Weight in α^*, lower than $\frac{1}{2}$, results in a price that is higher than the equilibrium price Y^*. In each experiment, we have conducted 100 rounds, with x_0 generated uniformly from [1, 5], and each round has 20,000 iterations. The mean $\bar{\mu}$ and the standard deviation $\bar{\sigma}$ of $\{x_k\}$ for each experiment are derived from the average sample of 100 rounds for iterations from $k = 10{,}000$ to $k = 20{,}000$

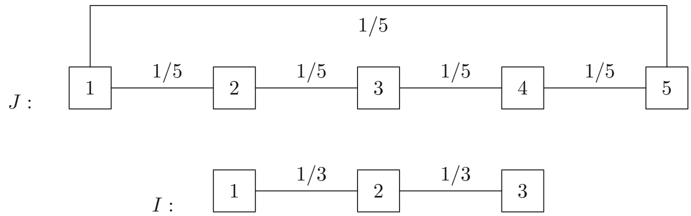

Fig. 2 Markovian networks of J and I and their transition probabilities

as two time non-homogeneous Markovian chains. An example of an economy with five buyers, $J = \{1, 2, \ldots, 5\}$, and three sellers, $I = \{1, 2, 3\}$, is given in Fig. 2.

Thus, for each network at time k, there are two transition probability matrices $P_I(k)$ and $P_J(k)$ over the states I and J, i.e.,

$$[P_I(k)]_{i,i'} = Prob\{s(k+1) = i' | s(k) = i\} \quad \forall i, i' \in I = \{1, 2, \ldots, m\}$$

and

$$[P_J(k)]_{j,j'} = Prob\{s(k+1) = j' | s(k) = j\} \quad \forall j, j' \in J = \{1, 2, \ldots, n\}.$$

The following assumption is given in Ram et al. (2009) and plays an important role to our results.

Assumption 3.1 (a) All diagonal entries of $P_I(k)$ and $P_J(k)$ are all positive for each k; (b) all positive entries in $P_I(k)$ and $P_J(k)$ are uniformly bounded away from zero, i.e., there exist constants $\eta_I > 0$ and $\eta_J > 0$ such that $[P_I(k)]_{i,i'} \geq \eta_I$ whenever $[P_I(k)]_{i,i'} > 0$ and $[P_J(k)]_{j,j'} \geq \eta_J$ whenever $[P_J(k)]_{j,j'} > 0$; (c) the two matrices $P_I(k)$

and $P_J(k)$ are both doubly stochastic for each k, i.e., the sum of entries in every row and every column of the two matrices is equal to one.

For $\alpha_\theta \in [0,1], \theta = 1,2,\ldots,r$, at time k, there is a transition probability matrix $P_R(k)$ over the states $R = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$. Because α_θ, $\theta = 1,2,\ldots,r$, are fixed, we also use $R = \{1,2,\ldots,r\}$ for notation convenience where $\theta \in R$ is used as a substitute for α_θ, i.e.,

$$[P_R(k)]_{\theta,\theta'} = Prob\{s(k+1) = \theta' | s(k) = \theta\} \quad \forall \theta, \theta' \in R = \{1,2,\ldots,r\}.$$

We assume that the transition probability matrix $P_R(k)$ satisfies the same assumption as in Assumption 3.1. Thus, the network linking all αs in R evolves across time.

Let $\Phi_I(k,\ell) = \Pi_{q=\ell}^{k-1} P_I(q)$, $\Phi_J(k,\ell) = \Pi_{q=\ell}^{k-1} P_J(q)$, and $\Phi_R(k,\ell) = \Pi_{q=\ell}^{k-1} P_R(q)$, with $k > \ell$, denote the three corresponding transition probability matrices for the Markov chains from time ℓ to k. Then, the convergence results for them under Assumption 3.1 are given by (Lemma 4.1 in Ram et al. 2009),

$$\left| [\Phi_I(k,\ell)]_{i,j} - \frac{1}{m} \right| \le b_I \beta_I^{k-\ell}, \quad \text{for all } k \text{ and } \ell \text{ with } k \ge \ell \ge 0 \tag{3.1}$$

$$\left| [\Phi_J(k,\ell)]_{i,j} - \frac{1}{n} \right| \le b_J \beta_J^{k-\ell}, \quad \text{for all } k \text{ and } \ell \text{ with } k \ge \ell \ge 0, \tag{3.2}$$

$$\left| [\Phi_R(k,\ell)]_{i,j} - \frac{1}{r} \right| \le b_R \beta_R^{k-\ell}, \quad \text{for all } k \text{ and } \ell \text{ with } k \ge \ell \ge 0, \tag{3.3}$$

where

$$b_I = \left(1 - \frac{\eta_I}{2m^2}\right)^{-2}, \quad \beta_I = \left(1 - \frac{\eta_I}{2m^2}\right)^{\frac{1}{Q_I}},$$
$$b_J = \left(1 - \frac{\eta_J}{2n^2}\right)^{-2}, \quad \beta_J = \left(1 - \frac{\eta_J}{2n^2}\right)^{\frac{1}{Q_J}},$$
$$b_R = \left(1 - \frac{\eta_R}{2r^2}\right)^{-2}, \quad \beta_R = \left(1 - \frac{\eta_R}{2r^2}\right)^{\frac{1}{Q_R}},$$

where $Q_I \ge 1$, $Q_J \ge 1$, and $Q_R \ge 1$ are some integers defined as follows. As in Ram et al. (2009), let $V_I = \{1,2,\ldots,m\}$ be the set of vertices for the seller network. There is an edge in the seller network $E_I(k)$ between sellers $i1$ and $i2$ at time k if and only if $[P(k)]_{i1,i2} > 0$. There exists an integer $Q_I \ge 1$ such that the graph $(V_I, \cup_{l=k}^{k+Q_I-1} E_I(l))$ is strongly connected for all k. One can similarly define Q_J and Q_R. The existence of Q_I or Q_J guarantees that an agent in I or J has a chance, within a finite number of price iterations, of getting into the price iteration process so that her private information will be reflected in $\{x_k\}$. Likewise, Q_R guarantees that each α in R will be eventually used in the iteration process of the auction.

Assumptions 3.1(a), (b) imply that each diagonal entry is uniformly bounded away from zero and one; and any positive non-diagonal entry is uniformly bounded from zero. Doubly stochastic assumption of Assumption 3.1(c) is important and somehow restrictive.

Under Assumption 3.1, Eq. (3.1) shows that the transition matrix $\Phi_I(k,l)$ from l to k can be approximated by a matrix with each entry that equals $\frac{1}{m}$; The error of such an approximation is bounded by $b_I\beta_I^{k-\ell}$. Equations (3.2) and (3.3) can be explained similarly.

4 Subgradients

Given a convex function φ: $R^d \to R$, a vector $\bigtriangledown\varphi \in R^d$ is a subgradient of φ at $y \in R^d$ if $\varphi(z) \geq \varphi(y) + \langle \bigtriangledown\varphi, z - y\rangle$ for all $z \in R^d$. The set of all subgradients of a convex function is called the subdifferential of φ at y, denoted $\partial\varphi(y)$. For a convex function φ, the subdifferential $\partial\varphi(y)$ is a nonempty, compact, and convex set for every $y \in R^d$ (see Clarke et al. 1988; Bertsekas 2009). Moreover, $\cup_{y\in Y}\partial\varphi(y)$ is bounded for a bounded set Y (Bertsekas 2009, p. 185).

For any two regular functions φ and ψ at y, the sum $\varphi + \psi$ is regular at y and satisfies

$$\partial(\varphi + \psi)(y) = \partial\varphi(y) + \partial\psi(y).$$

Let Y^* denote the set of solutions to $\mathcal{P}$. Let $dist(y, Y^*) = \inf_{y^*\in Y^*} \|y - y^*\|$, where $\|\cdot\|$ denotes the Euclidean norm. We may assume that the price space Y for $\mathcal{P}$ is a compact subset in R^d_+.

A large class of exchange economies for selling indivisible goods can be modeled by $\mathcal{P}$, including the assignment problem and an exchange economy with indivisible goods (see, e.g., Bikhchandani and Mamer 1997) typified by the job matching model of Kelso and Crawford (1982). Let D_j denote the demand curve for buyer $j \in J$ and S_i the supply curve for seller $i \in I$. Then, for a large class of quasilinear economies, we have $\bar{co}D_j(y) = -\partial g_j(y)$ and $\bar{co}S_i(y) = \partial f_i(y)$ for all $y \in Y$, where $\bar{co}A$ denotes the closed convex hull of A (see Ma and Nie 2003).

For an economy given in $\mathcal{P}$, the following holds

$$0 \in \sum_{i\in I} \bar{co}S_i(y) - \sum_{j\in J} \bar{co}D_j(y)$$

for all $y \in Y^*$, since $0 \in \sum_{i=1}^m \partial f_i(y) + \sum_{j=1}^n \partial g_j(y)$ for all $y \in Y^*$. For a class of exchange economies for selling indivisible goods typified by the job matching model of Kelso and Crawford (1982) with the gross substitutes condition, the following equality holds

$$0 \in \sum_{i\in I} S_i(y) - \sum_{j\in J} D_j(y)$$

for all $y \in Y^*$. Thus, Y^* is the set of Walrasian equilibrium prices. For this reason, we also say a solution to $\mathcal{P}$ a Walrasian equilibrium.

Recall $\alpha^* = \frac{1}{r}\sum_{\theta=1}^r \alpha_\theta$. Then $Y^* = Y^*(\alpha^*, \lambda)$ for $\lambda(1 - \alpha^*) = \alpha^*$. However, if the equality $\lambda(1 - \alpha^*) = \alpha^*$ does not hold, Y^* may be different from $Y^*(\alpha^*, \lambda)$.

5 Markovian α-double auctions

Let x_k be the price vector at time k, a vector of prices of the executed trade at time k. After observing the price vector x_k, a pair of seller and buyer $(w_k, w_k') \in I \times J$ is drawn according to some rule, specified in Sect. 3, to submit an ask order and a bid order to determine x_{k+1} of the next trade.

An ask order for seller w_k consists of a vector of ask prices $\psi_{w_k,k+1}$ and an ask size $(\bigtriangledown f_{w_k}(x_k) + \epsilon_{w_k,k})$, where $\bigtriangledown f_i(x_k)$ is the subgradient in $\partial f_i(x_k)$. The ask size is a vector of true quantities supplied $\bigtriangledown f_{w_k}(x_k)$ plus a vector of stochastic noises $\epsilon_{w_k,k}$ since f_{w_k} is known only partially to the seller w_k. The relationship between his ask prices and sizes is given by

$$\psi_{w_k,k+1} = x_k - a_k(\bigtriangledown f_{w_k}(x_k) + \epsilon_{w_k,k}), \tag{5.1}$$

where a_k is the ask step size. a_k is a time-varying price decrement for selling one unit (share). Individual selling puts a pressure for prices to move down. Moreover, the more the seller wants to sell the more will the prices move down.

Similarly, a bid order for buyer w_k' consists of a vector of bid prices $\varphi_{w_k',k+1}$ and a vector of bid sizes $(\bigtriangledown g_{w_k'}(x_k) + \epsilon_{w_k',k})$, where $\bigtriangledown g_{w_k'}(x_k)$ is the subgradient in $\partial g_{w_k'}(x_k)$ and $\epsilon_{w_k',k}$ is stochastic noises. The relationship between her bid prices and sizes is given by

$$\varphi_{w_k',k+1} = x_k - b_k\left(\bigtriangledown g_{w_k'}(x_k) + \delta_{w_k',k}\right), \tag{5.2}$$

where b_k is the bid step size. b_k is a time-varying price increment for buying one unit (share). Individual buying puts a pressure for prices to move up. Moreover, the more the buyer wants to buy the more will the prices move up.

The prices of the next executed trade x_{k+1} are a weighted average of ask and bid prices:

$$x_{k+1} = P_Y\left(\alpha_{w_k''}\psi_{k+1} + \left(1 - \alpha_{w_k''}\right)\varphi_{k+1}\right). \tag{5.3}$$

Here P_Y is the Euclidean projection onto Y. w_k, w_k' and w_k'' are respectively updating in time according to time non-homogeneous Markov chains with states $\{1, 2, \ldots, m\}$, $\{1, 2, \ldots, n\}$ and $\{1, 2, \ldots, r\}$, which satisfy Assumption 3.1. The auction from (5.1)–(5.3) follows that of Xu et al. (2014) with an exception that α is no longer fixed. Instead, it is a random variable governed by $P_R(k)$. This is a response to a question raised in Xu et al. (2014) that asks what will happen if α is at random, following some stochastic processes.

We can also write (5.1) and (5.2) in terms of demand and supply

$$\begin{aligned}\psi_{k+1} &= x_k - a_k\left(S_{w_k}(x_k) + \epsilon_{w_k,k}\right),\\ \varphi_{k+1} &= x_k + b_k\left(D_{w_k'}(x_k) - \delta_{w_k',k}\right),\end{aligned}$$

where $S_{w_k}(x_k) \in \bar{co}S_{w_k}(x_k) = \partial f_{w_k}(x_k)$ and $D_{w_k'}(x_k) \in \bar{co}D_{w_k'}(x_k) = -\partial g_{w_k'}(x_k)$. In fact, $S_{w_k}(x_k)$ and $D_{w_k'}(x_k)$ can be chosen without taking the closed convex hull when the traded goods are indivisible. This goes back to the familiar form given in (2.2) in the introduction because bid size equals $D_{w'_k}(x_k)$ plus some noise while ask size equals $S_{w_k}(x_k)$ plus some noise. We call this α-double auction on two Markovian networks α-MCDA. The sequence $\{x_k\}$ is called a Markovian price process.

An α-MCDA may be explained as follows. For the sale of a single stock with price x_k, starting at $l \geq 0$ such that $l < k$, a seller is drawn from the sellers' network according to the transition matrix $\Phi_I(k, l)$ and a buyer is drawn from the buyers' network according to the transition matrix $\Phi_J(k, l)$. The ask price equals the price x_k minus a linear increment of the number of shares for sale by the drawn seller and the bid price equals the price x_k plus a linear increment of the number of shares for purchase by the drawn buyer. The new updated price x_{k+1} is a weighted average of the new bid and ask prices, with weight that is drawn according to the transition matrix $\Phi_R(k, l)$. Moreover, the number of shares for sale and purchase may include stochastic noise variables. It is possible that the new updated price may be negative. In that case, by setting Y priorly to be a closed interval with a positive lower end price, any negative price will be replaced by the positive lower end of Y.

Since Y is compact, g is continuous, there exist scalars C, D and M such that

$$\|h\| \leq C, \quad \forall h \in \partial f_i(x_k), \ \ \forall i \in I, \ k = 0, 1, 2, \ldots, \tag{5.4}$$

$$\|\ell\| \leq D, \quad \forall \ell \in \partial g_j(x_k), \ \ \forall j \in J, k = 0, 1, 2, \ldots, \tag{5.5}$$

and

$$|g(y)| \leq M, \quad \forall y \in Y. \tag{5.6}$$

6 A simulation result

The following example has been studied in Xu et al. (2014). Now we use this example to illustrate how a time-varying α-double auction operates. The three sellers and five buyers have no knowledge at all where the Walrasian equilibrium locates. Neither do they know total demand and supply. Each of them submits the bid and ask orders based on self interest. It is the auction that integrates "dispersed bits of incomplete information" (Hayek 1945) into prices. This example demonstrates that a time-varying α-double auction can indeed reach the equilibrium of the original economy. However, this conclusion is based on a perfect scenario because it depends on key conditions on α and λ.

There are three sellers, $i = 1, 2, 3$, with each seller $i = 1, 2, 3$ an initial endowment of $(i + 1)$ units of an identical (divisible) good. There are five buyers $j = 1, 2, \ldots, 5$, each buyer j's consumer's surplus or profit function g_j: $R_+ \to R$ is obtained from

$$g_j(y) = \max_{q \geq 0} u_j(q) - qy,$$

where u_j: $[0, \infty) \to R_+$ is j's utility function given by $u_j(q) = (j+1) + 2\sqrt{(j+1)q}$. The supply curve for each seller is $S_i(y) = [0, i+1]$ for $y = 0$ and $S_i(y) = i+1$ for

$y>0$, $i=1,2,3$. The demand curve $D_j(y)=q_j^*$, where $u_i'(q_j^*)=y$ for $y \gg 0$, $j=1,2,\ldots,5$. In this example, we can set $f_i(y)=(i+1)y$ for $i \in I=\{1,2,3\}$ and $g_j(y)=(j+1)+\frac{j+1}{y}$ for $j \in J=\{1,2,\ldots,5\}$ so that $D_j(y)=q_j^*=\frac{j+1}{y^2}$. Thus, the equilibrium price equals $y^*=\sqrt{\frac{20}{9}}=1.49$.

The network structures of J and I and the transition probabilities are shown in Fig. 2. We set $R=\{0.2, 0.5, 0.8\}$ and $P_R(k)=P_I(k)$ for all k. Thus, $\alpha^*=0.5$. The two step sizes $\{a_k\}$ and $\{b_k\}$ satisfy $a_k=\frac{1}{k+1}$ and $b_k=\lambda\frac{5}{3}a_k$. In the case with noises, the Gauss noise (0, 25) is used for sellers and buyers, where 0 is the mean of the noise term while 25 is the variance. We provide a simulation with and without noises. The simulation results are shown in Fig. 3a–c for $\lambda=1$. Figure 3c

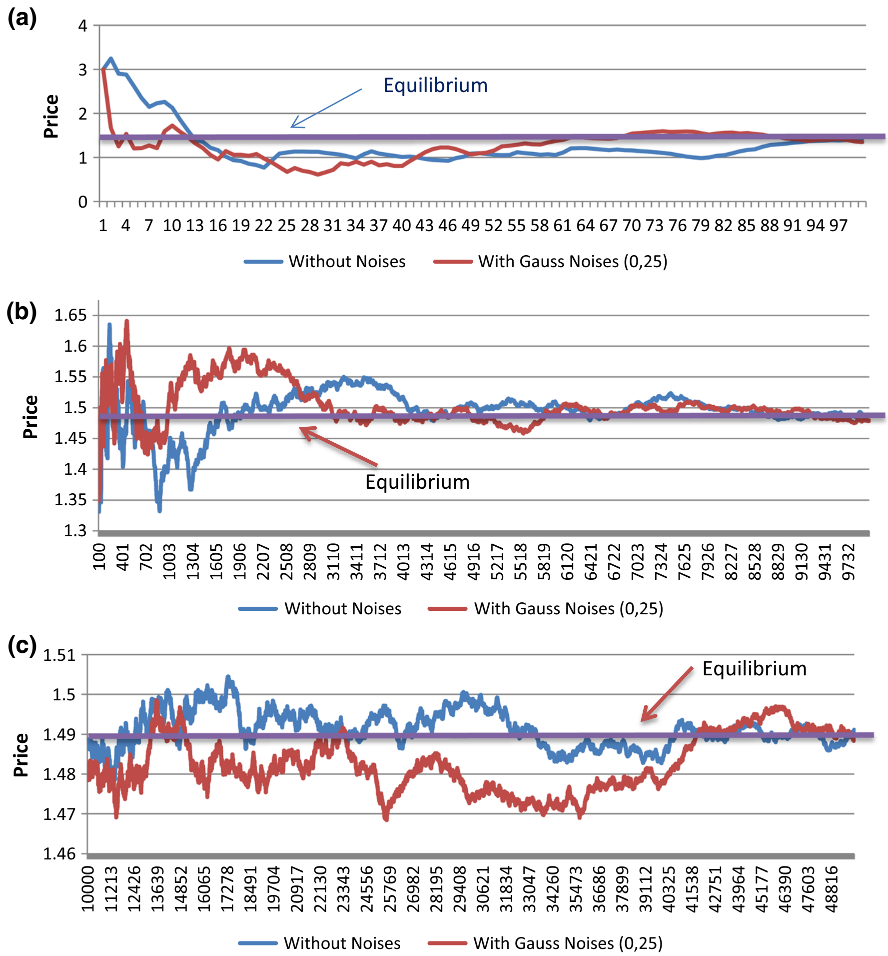

Fig. 3 Price process $\{x_k\}$, $x_0=3$, total iterations $=$ 50,000. Equilibrium price $=$ 1.49. **a** Sample is from 1 to 100. **b** Sample is from 100 to 10,000. **c** Sample is from 10,000 to 50,000

is the price charts near equilibrium or the limit of the price process and represents two trading price charts for a period of 11.11 h, with one trade per second (iteration). Theorem 7.1 shows that the price sequence $\{x_k\}$ converges to the equilibrium price 1.49, with probability 1, because $\alpha^* = 0.5$ and $\lambda = 1$. Note that these price sequences are not necessarily unique since there are transition probability matrices involved (see a robustness study with multiple rounds given in Sect. 8.5).

Convergence of an α-double auction to a Walrasian equilibrium is very subtle. When α is given, there is no positive scalar λ such that the equality, $\alpha = \lambda(1-\alpha)$, holds for either $\alpha = 0$ or 1. When α is time-varying as specified in this study, there always exists a set of states $R = \{\alpha_\theta | \theta = 1, 2, \ldots, r\}$ such that $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_r \le 1$ and the equality, $\alpha^* = \lambda(1-\alpha^*)$, holds for any positive scalar λ. This implies that a Markovian α-double auction has a major advantage over an α-double auction with a fixed α for the market searching for an equilibrium. This is especially important if the auction often executes trades near the two end values $\alpha = 1$ and $\alpha = 0$. Our main finding in this paper is this new α^*.

7 Main results

In the following, let G_k denote the entire history of the method up to time $k-1$, i.e. the σ-field generated by the initial vector x_0 and $\{w_i, w_i', w_i'', \epsilon_{w_i,i}, \delta_{w_i',i}; i = 0, 1, \ldots, k-1\}$. We make the following assumption:

Assumptions 7.1 There exist deterministic scalar sequences $\{\mu_k\}, \{\nu_k\}, \{\tau_k\}$ and $\{\sigma_k\}$ such that for $k = 1, 2, \ldots,$

$$||E[\epsilon_{w_k,k}|G_k]|| \le \mu_k, \quad ||E[\delta_{w_k',k}|G_k]|| \le \tau_k;$$
$$E[||\epsilon_{w_k,k}||^2|G_k] \le \nu_k^2, \quad E[||\delta_{w_k',k}||^2|G_k] \le \sigma_k^2.$$

Note that these noises are not necessarily Gaussian and they may persist all the way with the evolution of the price process. We consider these noises in order to capture the possibility buyers and sellers may have uncertainty about their true demand and supply when they submit their bid and ask orders, respectively. Different buyers and sellers may also have different degree of uncertainties about their demand and supply, respectively.

Lemma 7.1 *Let Assumptions 3.1 and 7.1 hold. Then the following inequality holds for the α-MCDA iteration (5.1)–(5.2) for any step size rule, any $y \in Y$ and $k = 0, 1, 2, \ldots$:*

$$
\begin{aligned}
&E\left[\|x_{k+1}-y\|^2 \middle| G_{d(k)}\right] \\
&\le E\left[\|x_k-y\|^2 \middle| G_{d(k)}\right] - \frac{2\alpha^* a_k}{m}\left(f(x_{d(k)})-f(y)\right) - \frac{2(1-\alpha^*)b_k}{n}\left(g(x_{d(k)})-g(y)\right) \\
&\quad + \left(2a_k rC\left(b_R\beta_R^{k+1-d(k)} + mb_I\beta_I^{k+1-d(k)}\right)\right. \\
&\quad \left. + 2b_k rD\left(b_R\beta_R^{k+1-d(k)} + nb_J\beta_J^{k+1-d(k)}\right)\right)\|x_{d(k)}-y\| \\
&\quad + 2(a_kC+b_kD)\sum_{\ell=d(k)}^{k-1}\left(a_\ell(C+\nu_\ell)+b_\ell(D+\sigma_\ell)\right) + (a_kC+a_k\nu_k+b_kD+b_k\sigma_k)^2 \\
&\quad + (2a_k\mu_k+2b_k\tau_k)E[\|x_k-y\||G_{d(k)}],
\end{aligned} \tag{7.1}
$$

where $\{d(k)\}$ is any non-negative integer sequence with $d(k) \le k$.

Proof See the "Appendix". □

Remark Lemma 7.1 plays a key role for proving our main results in Theorem 7.1 below. It shows that the quadratic distance of the price process to the equilibrium can be bounded and it is possible for the price process to get closer and closer to the equilibrium. Interestingly, the equilibrium point y chosen cannot be a solution in Y^*. Instead, it must be in $Y^*(\alpha^*, \lambda)$, as shown in Theorem 7.1.

Now we can prove our main results.

Theorem 7.1 *Let Assumptions 3.1 and 7.1 hold. Assume that the step sizes are of the forms*

$$a_k = \frac{A}{(k+1)^p}, \quad b_k = \frac{B}{(k+1)^q}$$

with A, B positive scalars, $\frac{2}{3} < p \le 1$ and $\frac{2}{3} < q \le 1$, that there exists some positive λ with

$$\sum_{k=0}^{\infty}\left|\frac{b_k}{n} - \lambda\frac{a_k}{m}\right| < +\infty \tag{7.2}$$

and that the following estimates hold:

$$\sum_{k=0}^{\infty} a_k\mu_k < \infty, \quad \sum_{k=0}^{\infty} b_k\tau_k < \infty, \quad \sup_{k\ge 0}\nu_k < \infty, \quad \sup_{k\ge 0}\sigma_k < \infty. \tag{7.3}$$

Then the following results hold with probability 1 for the sequence $\{x_k\}$ generated by α-MCDA algorithm (5.1), (5.2):

$$\liminf_{k\to\infty} F(x_k, \alpha^*, \lambda) = F^*(\alpha^*, \lambda), \quad \liminf_{k\to\infty} dist(x_k, Y^*(\alpha^*, \lambda)) = 0. \tag{7.4}$$

Further more, we have

$$\lim_{k\to\infty} E[dist(x_k, Y^*(\alpha^*, \lambda))^2] = 0. \tag{7.5}$$

Proof See the "Appendix". □

Remark The two step sizes follow Ram et al. (2009) because we need to use their result for $\sum_{k=2}^{\infty}\eta_k < \infty$ in the proof in the "Appendix". The supermartingale convergence theorem in Lemma 1 in Bertsekas and Tsitsiklis (2000) plays a critical role in our proof. Without condition (7.2), we cannot use this theorem because $\frac{2\alpha^* a_k}{m}(E[f(x_{d(k)})] - f(y^*)) + \frac{2(1-\alpha^*)b_k}{n}(E[g(x_{d(k)})] - g(y^*))$ that appears in inequality (10.9) in the proof of Theorem 7.1 may not be nonnegative. There is no such a problem if a single sequence of step sizes is used as in the incremental subgradient methods (Nedić and Bertsekas 2001; Ram et al. 2009).

Assumptions 7.1 and (7.3) follow Ram et al. (2009) again. Note that μ_k and τ_k cannot be set at nonzero constants for all k. Nonetheless, they may be very large, positive or negative, for any finite number of k. Moreover, μ_k and τ_k should be "small" but need not to be zero. Their variances can be large and set at constants as long as they are bounded. These conditions say that buyers or sellers may never place orders that are on their true demand or supply curves but the errors on the average should remain small.

Under the incremental subgradient method in Nedić and Bertsekas (2001) and Ram et al. (2009), the price processes always converge to a solution in Y^* for the diminishing step size rule. Our results in Theorem 7.1 show that such a result is no longer true, because Y^* may be very different from $Y^*(\alpha^*, \lambda)$. Xu et al. (2014) provided similar results in Theorem 7.1 for a fixed $\alpha \in [0, 1]$. When α follows a Markovian chain governed by Assumption 3.1, we need to replace the fixed α with the average α^* (firstly defined in the abstract).

Finally, we would like to point out that λ may not exist to satisfy condition (7.2) for some diminishing step sizes a_k and b_k given in Theorem 7.1. But, if it exists, it must be unique. Moreover, if the limit $\frac{b_k}{a_k}$ exists, then $\lambda = \frac{m}{n}\lim_{k\to\infty}\frac{b_k}{a_k}$. Even if the limit $\frac{b_k}{a_k}$ does not exist, there may exist λ to satisfy condition (7.2). Such a λ satisfies the condition $\frac{m}{n}\liminf_{k\to\infty}\frac{b_k}{a_k} < \lambda < \frac{m}{n}\limsup_{k\to\infty}\frac{b_k}{a_k}$. These points have been discussed in detail in Ma and Li (2011).

Since $Y^*(\alpha^*, \lambda) = Y^*$ for $\alpha^* = \lambda(1-\alpha^*)$, it follows from (7.5) that the price process $\{x_k\}$ converges in probability to the set of Walrasian equilibria of the original economy when $\alpha^* = \lambda(1-\alpha^*)$. If, in addition, Y^* is a singleton, then the price process $\{x_k\}$ converges in probability 1 to the Walrasian equilibrium of the original economy.

8 Discussions

The first part of the discussions is about the important role of equilibrium for the convergence of the Markovian α-double auction in Theorem 7.1. The second part focuses on the issue of investment sentiments. The third part considers the case where the condition on λ does not satisfy. We use the example in Sect. 6 and demonstrate that Theorem 7.1 does not hold if the condition fails. The fourth part explores two directions such that Assumption 3.1 may be relaxed for α. Our results are numerical in nature and instructional for theoretical researches in the future.

8.1 Gravitation of equilibrium

Smith (1776) clearly stated that there is a natural price for every commodity and the natural price of a commodity should be equal to its equilibrium price:

> When the quantity brought to market is just sufficient to supply the effectual demand and no more, the market price naturally comes to be either exactly, or as nearly as can be judged of, the same with the natural price. (Smith 1776, Book I.7.11)

Moreover, he predicted what may happen if the market price is not at the natural price:

> The natural price, therefore, is, as it were, the central price, to which the prices of all commodities are continually gravitating. Different accidents may sometimes keep them suspended a good deal above it, and sometimes force them down even somewhat below it. But whatever may be the obstacles which hinder them from settling in this center of repose and continuance, they are constantly tending towards it. (Smith 1776, Book I.7.15)

By following Adam Smith's idea of the natural price of a trading asset, we come up with a simple, but familiar, model

$$P_t^* = P_t + \epsilon_t$$

where P_t^* is the natural price of an asset at time t or the equilibrium price of an underlying economy. P_t is the market or trading price under the clearinghouse of an exchange. The reason we need a clearinghouse to find the natural price of an asset is that the effectual demand or supply for an asset is unknown. Each individual on the demand or supply side may know some private information about the asset but she is unlikely to know the total or effectual demand or supply. A clearinghouse is designed to find P_t^* via P_t. According to Adam Smith, the noise term ϵ_t must have mean zero and contain no information about P_t^*.

If P_t^* is seen as the natural price of an asset, it is still not easy to test if the efficient markets hypothesis is true because P_t^* must be modeled right in order to test the random walk theory (Fama 1965, 1991). In our study of an exchange economy, we may define the natural price of an asset to be the equilibrium price as defined in problem $\mathcal{P}$. Then, we can study the convergence of the price process P_t (in vectors for multiple assets). That is an advantage in the study of the double auctions in this paper because it bypasses the joint hypothesis problem articulated in Fama (1991).

Our main theorems show that a Markovian α-double auction can indeed find this natural price P_t^*, with probability 1, for an asset in a class of exchange economies.

An immediate warning is that such a conclusion puts a restrictive condition on the two parameters α^* and λ. Because the condition may not be satisfied in practice, the market price P_t may deviate from P_t^* to form bubbles and crashes.[3]

In practice, if P_t^* is seen as the present value of ex post dividends discounted with a constant discount rate, P_t may deviate from P_t^* in the form of excess volatility discovered by Shiller (1981) and LeRoy and Porter (1981). Our main results also provide conditions on how P_t may deviate from P_t^* for a sustainable period under a Markovian α-double auction. The condition to create bubbles or crashes is the equality, $\alpha^* = \lambda(1 - \alpha^*)$, does not hold. If this condition is not satisfied for a sustainable period, the price process $\{P_t\}$ still converges, but it may not converge to P_t^*. Thus, our results show that a double auction as a clearinghouse may contribute to the excess volatility. Bubbles and crashes are also observed in experimental double auctions for selling assets (see, e.g., Smith et al. 1988), but these bubbles or crashes can just last for a short period due to time and space limitations or budget concerns in experiments.

Hayek (1945) raised a closely related question as Smith (1776) did on how a general price mechanism in a laissez-faire economy finds the natural price of a commodity or an asset. von Hayek focused more on the information integration. He wonders how individual participants act solely in their self-interests without any guidance from a central authority in a marketplace to integrate "dispersed bits of incomplete information" correctly into the prices to find the natural prices of all commodities. He utilized the word "marvel" to shock the reader for its significance of the market mechanism and stated: "this mechanism would have been acclaimed as one of the greatest triumphs of the human mind[$\cdots$] [if] it were the result of deliberate human design" (Hayek 1945, p. 527).

"Dispersed bits of incomplete information" have been integrated into prices through trading between sellers and buyers, one pair at a time, in a double auction institution. The same question asked by von Hayek also applies to the double auction mechanism. Our studies of this Markovian α-double auction make a small step forward in answering his question. Extensive research must be done in order to answer his question affirmatively and satisfactorily.

8.2 Sentiments matter

Theorem 7.1 requires the two step sizes $a_k = \frac{A}{(k+1)^p}$ and $b_k = \frac{B}{(k+1)^q}$ satisfy the conditions $\frac{2}{3} < p, q \leq 1$. Because step sizes a_k is the price decrement per unit by the seller in the ask and b_k is the price increment per unit by the buyer in the bid, the two scalars A and B provide two measures for how aggressively the sellers and the buyers place their orders. Therefore, sentiments such as fear and greed may also be a factor in A and B.

[3] Bubbles and crashes in economics and finance are the price movements of an asset that are not supported by its fundamentals. Changes in fundamentals that result in higher or lower prices are not seen as bubbles or crashes. But sentiments that over or under-react to the news in fundamentals can lead to a formation of a bubble and a crash.

The condition on λ

$$\sum_{k=0}^{\infty}\left|\frac{b_k}{n}-\lambda\frac{a_k}{m}\right|<+\infty \tag{8.1}$$

in Theorem 7.1 implies that $p=q$ and

$$\lambda=\frac{m}{n}\frac{B}{A}. \tag{8.2}$$

To reach a Walrasian equilibrium for the Markovian α-double auction, Theorem 7.1 also shows that α^* must satisfy

$$\alpha^*=\frac{\lambda}{1+\lambda}. \tag{8.3}$$

In a well-functioning market, we may assume that m equals n. Then $\lambda=\frac{B}{A}$. When the market is neutral, neither bullish nor bearish, we expect that $B=A$ and then $\lambda=1$. Thus, $\alpha^*=0.5$. Under such a perfect scenario, Theorem 7.1 shows that the Markovian α-double auction converges to a Walrasian equilibrium.

Now consider what may occur when the market becomes more bullish or when the greed dominates, we can expect that $B>A$ and then $\lambda>1$. In a bull market, we should expect that executed trades are settled more often near the end side of the bids. Therefore, α^* should be biased[4] towards to a value less than 0.5, but this will upset Eq. (8.3). Thus, we can conclude that the Markovian α-double auction will converge to a price that is above its equilibrium price for a stock when the sentiment for it is bullish. Note that such a change in the sentiment does not cause any change in fundamentals because all f_i and g_j stay exactly the same. Why does sentiment matter? It is because the two parameters α^* and λ affect P_t^*. In our example in Sect. 6, P_t converges to $\sqrt{\lambda\frac{1-\alpha^*}{\alpha^*}}P^*$ not P^* under the Markovian α-double auction. The gravitation of the equilibrium of Smith (1776) still works but the natural price has been distorted in the process of searching for it.

To see the importance of $\alpha^*=\lambda(1-\alpha^*)$, we let $R=\{0.1,0.2,0.3\}$, a "bull" market with $\alpha^*=0.2$, and $R=\{0.7,0.8,0.9\}$, a "bear" market with $\alpha^*=0.8$. Then we conduct simulations with the example in Sect. 6, with the same $P_R(k)$, $k=0,1,2,\ldots$, and step sizes. We also let $\lambda=1$ and the noises are still Gauss noises (0, 25). For the case with $\alpha^*=0.2$, our price process should converge to the new equilibrium $\sqrt{4}\times 1.49$, 100 % higher than 1.49; and for the case with $\alpha^*=0.8$, our price process should converge to the new equilibrium $\frac{1}{\sqrt{4}}\times 1.49$, 50 % lower than 1.49. Figure 4a, b shows these simulation results and our two price processes perform exactly as expected. Clearly, the asset is overvalued for the case $\alpha^*=0.2$

[4] In practice, an execution of a trade is complicated. Given a quote (bid, ask) = (200, 201) for a stock in a dealer's market, the ask 201 is the best price available from the dealer to sell when a buyer places a market order to buy and the bid 200 is the best offer available from the dealer to buy when a seller places a market order to sell. The actual posted price may often lie within (200, 201). In a "bull" market for a stock where its price goes up more often than down, the actual executed price is likely closer to the ask price of the dealer.

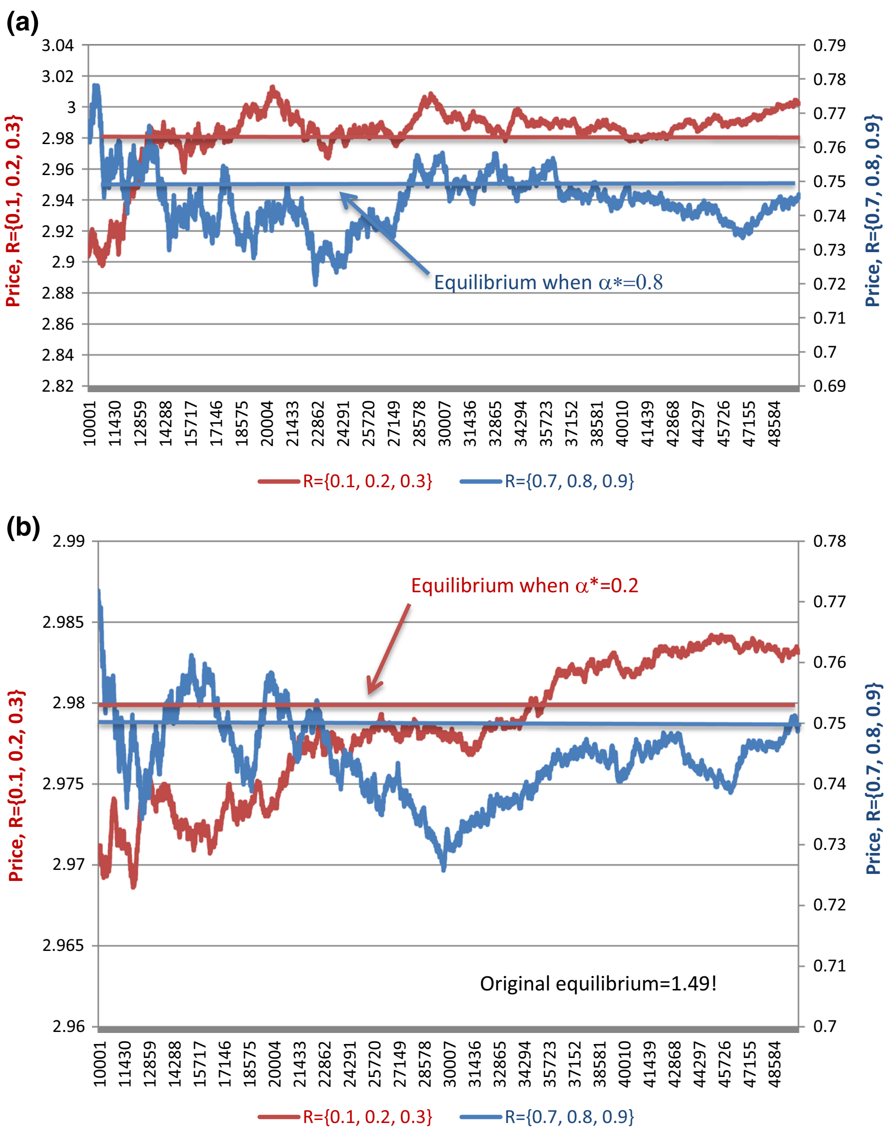

Fig. 4 The price processes $\{x_k\}$, with different α^*, $x_0 = 3$, total iterations = 50,000. Sample is from 10,000 to 50,000. **a** Noises (0, 25). **b** Without noises

and undervalued for the case $\alpha^* = 0.8$. An asset may be overvalued to an enormous level. For example, if $\alpha^* = 0.01$, then the price process will converge to 14.83, about ten times as large as the original equilibrium price 1.49. This indicates that the Markovian α-double auction may provide a useful workhorse to study the very issue related to the excess volatility in Shiller (1981).

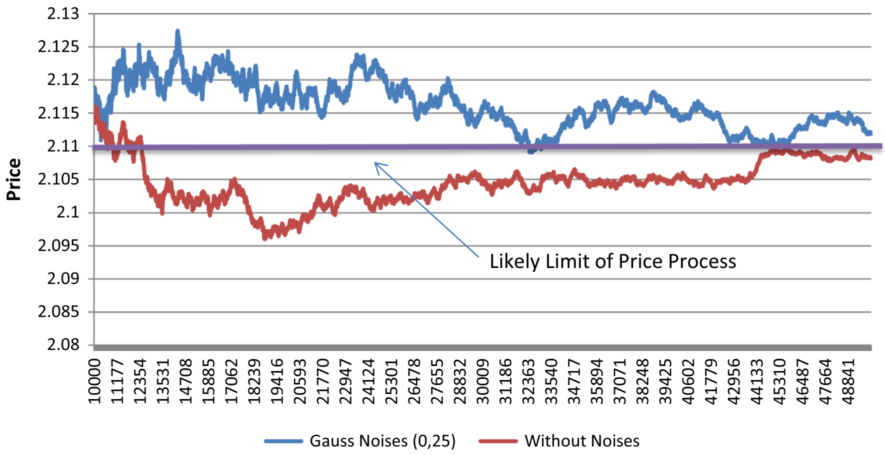

Fig. 5 Price process $\{x_k\}, x_0 = 3$, total iterations = 50,000. Sample is from 10,000 to 50,000. Equilibrium price = 1.49

8.3 Condition on λ fails

What may occur if the condition on λ in (7.2) fails to hold? To understand that case, we set $a_k = \frac{1}{k+1}$, $b_k = \lambda \frac{10}{3} a_k$, and $\lambda = 1$ in the example in Sect. 6, while $P_J(k)$ and $P_I(k)$ are the same for $k = 0, 1, 2, \ldots$, given in Fig. 2. Thus, the only difference here from the simulations in Sect. 6 is that the condition (7.2) on $\lambda > 0$ is no longer satisfied. We also set α at 0.5. We conduct a simulation for 50,000 iterations and find that the Markovian α-double auction converges to a price near 2.11, with Gauss noises (0, 25) and without noises. Because there is no convergence result available in theory, the price limit is our reading from our simulation results and subject to errors. The simulation results for sample from 10,000 to 50,000, with Gauss noises (0, 25) and without noises, have been shown in Fig. 5. From Fig. 5, we observe that the price process still converges, with probability 1, but it fails to converge to the equilibrium 1.49. The limit of the convergence is about 42 % higher than the equilibrium price 1.49. We do not have an explanation for the forces that lead to such a bubble. The sentiment argument above may provide a good explanation but the theorems there depend on the condition on λ to be satisfied. Interestingly, the price chart in Fig. 5 near the limit, once again, resembles amazingly the daily trading charts in a real stock market.

Our simulation here reveals that even if the equality, $\alpha^* = \lambda(1 - \alpha^*)$, holds, a failure of condition (7.2) can also result in a deviation from the equilibrium under the Markovian α-double auction, a case upsetting what had been predicted by Smith (1776) about the gravitation of equilibrium.

8.4 Alternative conditions on α

In Sect. 3, there are two key assumptions on the α network about R. Assumption 3.1 requires that the transition probability matrix $P_R(k)$ be doubly stochastic and the α

network be strongly connected. We now use the example in Sect. 6 to do two simulations and these simulation results show that these two conditions may be relaxed. In the first simulation, we let $R = \{0.2, 0.5, 0.8\}$, and the corresponding transition probability matrix is given by

$$P_R(k) = \begin{pmatrix} 0.7 & 0 & 0.3 \\ 0 & 1 & 0 \\ 0.3 & 0 & 0.7 \end{pmatrix}$$

for all k. The step sizes are given by

$$a_k = \frac{1}{k+1} \qquad b_k = \frac{5\lambda a_k}{3}$$

with $\lambda = 1$. In the case with noises, we still add Gauss noises with mean and variance (0, 25). The only difference here from Fig. 3 is a change in $P_R(k)$, which violates the strongly connection assumption in Sect. 3 (see Assumption 4 in Ram et al. 2009). The simulation results with and without noises are shown in Fig. 6. It appears that the two price charts converge, with probability 1, to the equilibrium 1.49. Of course, this convergence result is based on simulations, without a general result in theory. However, the two simulations do provide a direction how the strongly connection condition on R may be relaxed.

Another possible direction to extend Theorem 7.1 is to consider absorbing Markovian chains on R. In the next simulation, we also let $R = \{0.5, 0.2, 0.8\}$, but the corresponding transition probability matrix is changed to

$$P_R(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0.1 & 0.3 & 0.6 \\ 0.2 & 0.6 & 0.2 \end{pmatrix}$$

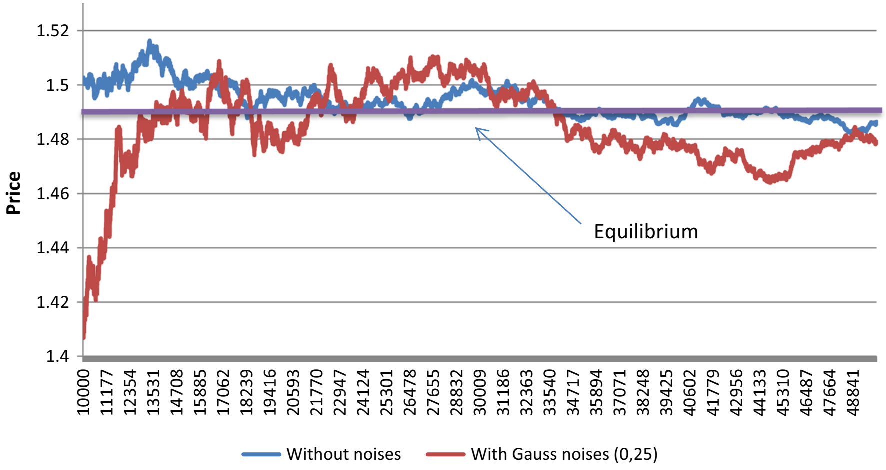

Fig. 6 Price process $\{x_k\}, x_0 = 3$, total iterations = 50,000. Sample is from 10,000 to 50,000. Equilibrium price = 1.49

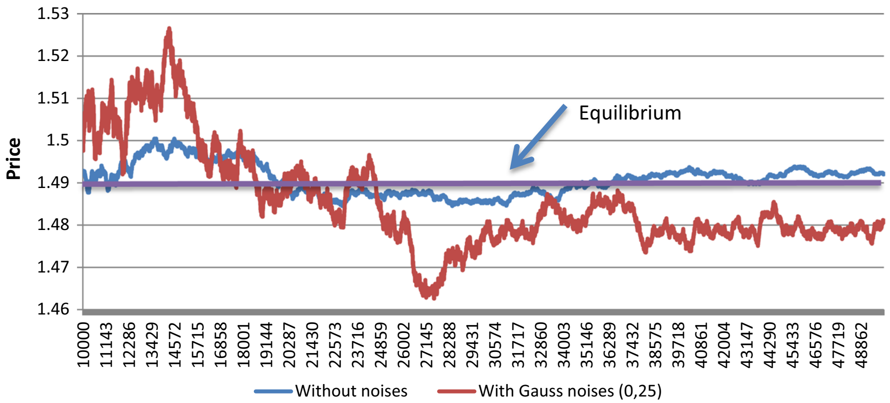

Fig. 7 Price process $\{x_k\}, x_0 = 3$, total iterations = 50,000. Sample is from 10,000 to 50,000. Equilibrium price = 1.49

for all k, which violates the doubly stochastic condition in Assumption 3.1. The step sizes remain the same as above. We also set $\lambda = 1$ and Gauss noises (0, 25) are used in the case with noises. Our simulation results are shown in Fig. 7. The two price processes suggest that they still converge, with probability 1, when $\alpha = 0.5$ is the absorbing state. But the convergence to equilibrium must be proved in theory to be conclusive for a claim if our results can be extended to Markovian chains with an absorbing state. It is also interesting to consider a general case where α is a random variable or follows a Brownian motion process.

8.5 Robustness

The excess volatility in Shiller (1981) shows that stock prices can substantially deviate from its fundamental (equilibrium) value of discounted dividends (with a constant discount factor). It is not easy to answer the question why there exists an excess volatility in a stock or a stock index. Our main theorem (Theorem 7.1) shows that a double auction, as a clearinghouse for stocks traded in an exchange, may contribute to the excess volatility. We have used a number of experiments to demonstrate that claim, see, e.g., Fig. 4a, b, where our experiments are done for one single round and the initial point x_0 is fixed at 3. This raises a question how robust our experiments may be, as pointed out by a referee. Next we answer this question by running several additional experiments with multiple rounds. At each round, we select the initial point x_0 uniformly from [1, 5] and conduct 20,000 iterations. We use the same example in Sect. 6 for various numbers of sellers and buyers, by setting $n = 5, 10, 20$ and $m = 5, 10, 20$, with and without noises. In that example, we know that the set of equilibrium prices Y^* is a singleton consisting of $\sqrt{\frac{n(n+3)}{m(m+3)}}$. But according to our Theorem 7.1, the price process $\{x_k\}$ converges in probability 1 to the set of equilibrium prices $Y^*(\alpha^*, \lambda)$, also a singleton with $\sqrt{\frac{\lambda(1-\alpha^*)}{\alpha^*}}\sqrt{\frac{n(n+3)}{m(m+3)}}$. This implies that the price process $\{x_k\}$ converges to the equilibrium price in Y^* when $\sqrt{\frac{\lambda(1-\alpha^*)}{\alpha^*}} = 1$, a bubble price (higher than the equilibrium one) when $\sqrt{\frac{\lambda(1-\alpha^*)}{\alpha^*}} > 1$, and a crash price (lower than the equilibrium one) when $\sqrt{\frac{\lambda(1-\alpha^*)}{\alpha^*}} < 1$. Our multiple round experiments confirm these results, implied by our theorem. We have conducted 18 additional experiments and 100 rounds for each experiment. We also let $Y = [0.01, 10]$. We will report the average results of the 100 rounds of each experiment in two tables and a number of figures. Tables 1 and 2 are below. The detail of these experiments and the figures are in "Experiments for Sect. 8.5" section of Appendix. The 100 round experimental data for each case are available upon request.

Tables 1 and 2 provide some useful statistics of these experiments. In the two tables, Y^* is the equilibrium price of the original economy. The price process $\{x_k\}$ converges to $Y^*(\alpha^*, 1)$ with probability 1, according to Theorem 7.1, depending on the average value of the weight α^*. Thus, a change in the average α^* affects the convergence of the price process $\{x_k\}$ of an α-double auction. These experiments provide solid evidence for our claim that a double auction implemented in a real exchange market may indeed contribute to the excess volatility.

9 Conclusions

This paper studies an α-double auction under which buyers and sellers form two time non-homogeneous Markovian chains with transition probability matrices specified in Ram et al. (2009) and Xu et al. (2014), with α evolving with time in a way governed by a time non-homogeneous Markovian chain as well. As in Xu et al. (2014), we find that the parameter λ, which may be seen as a measure for how aggressively buyers and sellers submit their bids and asks, plays an important role for the convergence of a Markovian α-double auction. For a large class of exchange economies with multiple indivisible goods (e.g., Bikhchandani and Mamer 1997, among others), typified by Kelso and Crawford's (1982) many-to-one job matching market with the gross substitutes condition, we provide conditions so that the price process generated by the Markovian α-double auction converges to the set of Walrasian equilibria of the original economy.

According to Smith (1776), an asset's price must be determined by its fundamental or natural price. The problem is that individual demands and supplies for an asset are privately known and information about them is dispersed and incomplete (Hayek 1945) to individuals. Under such a circumstance, majority real exchange markets across the world use a version of double auctions to discover the equilibrium price of an asset. Such an auction is not perfect because a price process under a double auction may converge to a price that is below or above the equilibrium price. Nonetheless, we also show that under certain reasonable conditions on the step sizes, the α-double auction can indeed find the equilibrium price, with contaminated individual demands and supplies.

We also provide a number of simulations. Our simulation results of a simple exchange economy with a single divisible good demonstrate that the price process of the Markovian α-double auction near equilibrium is very similar to a price process in a real exchange market. This provides evidence that the Markovian α-double auction may provide a useful framework for modeling a real exchange market. In particular, it may be used to study the excess volatility in the equity market. Indeed, our simulations provide examples to show that there are other forces beyond the equilibrium, in a market using double auctions, that can keep the price of an asset away from its natural or equilibrium price (see Tables 1, 2).

Our work follows the literature of incremental subgradient methods, widely used in a large scale of distributional computation, see Kibardin (1980), Bertsekas (2010), Nedić and Bertsekas (2001), and Ram et al. (2009), among others. Our Markovian α-double auction inherits that nice feature. It can apply to a market with a large number of buyers and sellers because each iteration of price updating in the Markovian α-double auction just involves a pair of a single buyer and a single seller. Information about the total demand or supply is not needed for its operation, a useful feature for a large scale economy. Our simulation example is quite limited in scale and it should be expanded to a very large scale. In particular, the number of buyers and the number of sellers can both be randomly generated. Such a study of the Markovian α-double auction will move in a leap step closer to a real exchange market.

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Compliance with ethical standards

Ethical standard This study was funded by Major Research Plan of National Science Foundation of China (91430105).

Conflict of interest The authors declare that they have no conflict of interest.

Appendix: Proofs of Lemma 7.1 and Theorem 7.1

Proof of Lemma 7.1

Proof The proof is similar to that of Lemma 4.2 in Ram et al. (2009). In fact, from the iteration (5.1)–(5.2), the "non-expansion" property of the Euclidean projection and the definition of subgradient, we have

$$
\begin{aligned}
||x_{k+1}-y||^2 &= \left|\left|P_Y\left(\alpha_{w_k''}\psi_{k+1}+\left(1-\alpha_{w_k''}\right)\varphi_{k+1}\right)-y\right|\right|^2\\
&= \left|\left|P_Y\left(x_k-\alpha_{w_k''}a_k\bigtriangledown f_{w_k}(x_k)-\alpha_{w_k''}a_k\epsilon_{w_k,k}-(1-\alpha_{w_k''})b_k\bigtriangledown g_{w_k'}(x_k)-\left(1-\alpha_{w_k''}\right)b_k\delta_{w_k',k}\right)-y\right|\right|^2\\
&\le \left|\left|(x_k-y)-\left(\alpha_{w_k''}a_k\bigtriangledown f_{w_k}(x_k)+\alpha_{w_k''}a_k\epsilon_{w_k,k}+(1-\alpha_{w_k''})b_k\bigtriangledown g_{w_k'}(x_k)+\left(1-\alpha_{w_k''}\right)b_k\delta_{w_k',k}\right)\right|\right|^2\\
&\le ||x_k-y||^2-2\alpha_{w_k''}a_k(x_k-y)^T\bigtriangledown f_{w_k}(x_k)-2\left(1-\alpha_{w_k''}\right)b_k(x_k-y)^T\bigtriangledown g_{w_k'}(x_k)\\
&\quad+\left|\left|\alpha_{w_k''}a_k\bigtriangledown f_{w_k}(x_k)+\alpha_{w_k''}a_k\epsilon_{w_k,k}+\left(1-\alpha_{w_k''}\right)b_k\bigtriangledown g_{w_k'}(x_k)+\left(1-\alpha_{w_k''}\right)b_k\delta_{w_k',k}\right|\right|^2\\
&\quad-\left(2\alpha_{w_k''}a_k(x_k-y)^T\epsilon_{w_k,k}+2\left(1-\alpha_{w_k''}\right)b_k(x_k-y)^T\delta_{w_k',k}\right)\\
&\le ||x_k-y||^2-2\alpha_{w_k''}a_k(f_{w_k}(x_k)-f_{w_k}(y))-2\left(1-\alpha_{w_k''}\right)b_k\left(g_{w_k'}(x_k)-g_{w_k'}(y)\right)\\
&\quad+\left|\left|\alpha_{w_k''}a_k\bigtriangledown f_{w_k}(x_k)+\alpha_{w_k''}a_k\epsilon_{w_k,k}+\left(1-\alpha_{w_k''}\right)b_k\bigtriangledown g_{w_k'}(x_k)+\left(1-\alpha_{w_k''}\right)b_k\delta_{w_k',k}\right|\right|^2\\
&\quad-\left(2\alpha_{w_k''}a_k(x_k-y)^T\epsilon_{w_k,k}+2\left(1-\alpha_{w_k''}\right)b_k(x_k-y)^T\delta_{w_k',k}\right).
\end{aligned}
\tag{10.1}
$$

Taking conditional expectations with respect to $G_{d(k)}$, we get

$$
\begin{aligned}
E\left[||x_{k+1}-y||^2\middle|G_{d(k)}\right] &\le E\left[||x_k-y||^2\middle|G_{d(k)}\right]-2a_kE\left[\alpha_{w_k''}(f_{w_k}(x_k)-f_{w_k}(y))\middle|G_{d(k)}\right]\\
&\quad-2b_kE\left[\left(1-\alpha_{w_k''}\right)\left(g_{w_k'}(x_k)-g_{w_k'}(y)\right)\middle|G_{d(k)}\right]\\
&\quad+E\left[\left|\left|\alpha_{w_k''}a_k\bigtriangledown f_{w_k}(x_k)+\alpha_{w_k''}a_k\epsilon_{w_k,k}+\left(1-\alpha_{w_k''}\right)b_k\bigtriangledown g_{w_k'}(x_k)+\left(1-\alpha_{w_k''}\right)b_k\delta_{w_k',k}\right|\right|^2\middle|G_{d(k)}\right]\\
&\quad-E\left[\left(2\alpha_{w_k''}a_k(x_k-y)^T\epsilon_{w_k,k}+2\left(1-\alpha_{w_k''}\right)b_k(x_k-y)^T\delta_{w_k',k}\right)\middle|G_{d(k)}\right]\\
&\le E\left[||x_k-y||^2\middle|G_{d(k)}\right]-2a_kE\left[\alpha_{w_k''}\left(f_{w_k}(x_k)-f_{w_k}(x_{d(k)})\right)\middle|G_{d(k)}\right]\\
&\quad-2a_kE\left[\alpha_{w_k''}\left(f_{w_k}(x_{d(k)})-f_{w_k}(y)\right)\middle|G_{d(k)}\right]-2b_kE\left[\left(1-\alpha_{w_k''}\right)\left(g_{w_k'}(x_k)-g_{w_k'}(x_{d(k)})\right)\middle|G_{d(k)}\right]\\
&\quad-2b_kE\left[\left(1-\alpha_{w_k''}\right)\left(g_{w_k'}(x_{d(k)})-g_{w_k'}(y)\right)\middle|G_{d(k)}\right]\\
&\quad+E\left[\left|\left|\alpha_{w_k''}a_k\bigtriangledown f_{w_k}(x_k)+\alpha_{w_k''}a_k\epsilon_{w_k,k}+\left(1-\alpha_{w_k''}\right)b_k\bigtriangledown g_{w_k'}(x_k)+\left(1-\alpha_{w_k''}\right)b_k\delta_{w_k',k}\right|\right|^2\middle|G_{d(k)}\right]\\
&\quad-E\left[\left(2\alpha_{w_k''}a_k(x_k-y)^T\epsilon_{w_k,k}+2\left(1-\alpha_{w_k''}\right)b_k(x_k-y)^T\delta_{w_k',k}\right)\middle|G_{d(k)}\right].
\end{aligned}
\tag{10.2}
$$

Now, we will estimate the second term to the seventh term in the second "$\le$" of (10.2) respectively. For the second term , by the definition of subgradient, subgradient boundedness (5.4) and the fact that $\alpha_{w_k''}\in[0,1]$, we have

$$
\begin{aligned}
&E\Big[\alpha_{w_k''}\big(f_{w_k}(x_k)-f_{w_k}(x_{d(k)})\big)|G_{d(k)}\Big]\geq E\Big[\alpha_{w_k''}\bigtriangledown f_{w_k}(x_{d(k)})^T(x_k-x_{d(k)})\big|G_{d(k)}\Big]\\
&\quad\geq -CE[||x_{d(k)}-x_k|||G_{d(k)}]\geq -C\sum_{\ell=d(k)}^{k-1}E[||x_{\ell+1}-x_\ell|||G_{d(k)}]\geq -C\sum_{\ell=d(k)}^{k-1}\\
&\quad E\Big[\Big|\Big|\alpha_{w_\ell''}a_\ell\bigtriangledown f_{w_\ell}(x_\ell)+\alpha_{w_\ell''}a_\ell\epsilon_{w_\ell,\ell}+\Big(1-\alpha_{w_\ell''}\Big)b_\ell\bigtriangledown g_{w_\ell'}(x_\ell)+\Big(1-\alpha_{w_\ell''}\Big)b_\ell\delta_{w_\ell',\ell}\Big|\Big||G_{d(k)}\Big]\\
&\quad\geq -C\sum_{\ell=d(k)}^{k-1}(a_\ell(C+v_\ell)+b_\ell(D+\sigma_\ell)).
\end{aligned}
\tag{10.3}
$$

Similarly, we can obtain

$$
E\Big[\Big(1-\alpha_{w_k''}\Big)(g_{w_k'}(x_k)-g_{w_k'}(x_{d(k)})|G_{d(k)})\Big]\geq -D\sum_{\ell=d(k)}^{k-1}(a_\ell(C+v_\ell)+b_\ell(D+\sigma_\ell)).
\tag{10.4}
$$

As for the third term in the second " $\leq$ " of (10.2), noting that $G_{d(k)}$ denotes the entire history of the method up to time $d(k)-1$, and the probability transition matrices for the Markov chains $\{w_k\}$, $\{w_k''\}$ from time $d(k)-1$ to k are $\Phi_I(k,d(k)-1)$ and $\Phi_R(k,d(k)-1)$ respectively, we have

$$
\begin{aligned}
&E\Big[\alpha_{w_k''}\big(f_{w_k}(x_{d(k)})-f_{w_k}(y)\big)|G_{d(k)}\Big]\\
&\quad=\sum_{t=1}^{r}\sum_{i=1}^{m}[\Phi_R(k,d(k)-1)]_{w''_{d(k)},t}[\Phi_I(k,d(k)-1)]_{w_{d(k)},i}\alpha_t\big(f_i(x_{d(k)})-f_i(y)\big)\\
&\quad\geq\sum_{t=1}^{r}[\Phi_R(k,d(k)-1)]_{w''_{d(k)},t}\alpha_t\sum_{i=1}^{m}\frac{1}{m}\big(f_i(x_{d(k)})-f_i(y)\big)\\
&\qquad-\sum_{t=1}^{r}[\Phi_R(k,d(k)-1)]_{w''_{d(k)},t}\alpha_t\sum_{i=1}^{m}|\Phi_I(k,d(k)-1)]_{w_{d(k)},i}-\frac{1}{m}||f_i(x_{d(k)})-f_i(y)|\\
&\quad\geq\sum_{t=1}^{r}[\Phi_R(k,d(k)-1)]_{w''_{d(k)},t}\alpha_t\sum_{i=1}^{m}\frac{1}{m}\big(f_i(x_{d(k)})-f_i(y)\big)\\
&\qquad-r\sum_{i=1}^{m}|\Phi_I(k,d(k)-1)]_{w_{d(k)},i}-\frac{1}{m}||f_i(x_{d(k)})-f_i(y)|\\
&\quad\geq\sum_{t=1}^{r}\left(\frac{1}{r}+\left([\Phi_R(k,d(k)-1)]_{w''_{d(k)},t}-\frac{1}{r}\right)\right)\frac{\alpha_t}{m}\big(f(x_{d(k)})-f(y)\big)\\
&\qquad-rb_I\beta_I^{k+1-d(k)}mC||x_{d(k)}-y||\\
&\quad\geq\frac{\alpha^*}{m}\big(f(x_{d(k)})-f(y)\big)-rC\Big(b_R\beta_R^{k+1-d(k)}+mb_I\beta_I^{k+1-d(k)}\Big)||x_{d(k)}-y||,
\end{aligned}
\tag{10.5}
$$

where in the second " $\geq$ ", we have used the fact that $[\Phi_R(k, d(k)-1)]_{w''_{d(k)},t} \leq 1$ and $\alpha_t \in [0,1]$, in the third " $\geq$ ", we have used (3.1), and in the last " $\geq$ ", we have used (3.3). Similarly, it holds

$$
\begin{aligned}
&E\left[\left(1-\alpha_{w''_k}\right)\left(g_{w'_k}(x_{d(k)})-g_{w'_k}(y)\right)|G_{d(k)}\right]\\
&\quad\geq \frac{1-\alpha^*}{n}\left(g(x_{d(k)})-g(y)\right)-rD\left(b_R\beta_R^{k+1-d(k)}+nb_J\beta_J^{k+1-d(k)}\right)||x_{d(k)}-y||.
\end{aligned}
\tag{10.6}
$$

For the sixth term in the second " $\leq$ " of (10.2), from Assumption 7.1, the boundedness (5.4)–(5.5) and the fact that $\alpha_{w''_k} \in [0,1]$, it follows

$$
\begin{aligned}
&E\left[\left(\alpha_{w''_k}a_k\nabla f_{w_k}(x_k)+\alpha_{w''_k}a_k\epsilon_{w_k,k}+\left(1-\alpha_{w''_k}\right)b_k\nabla g_{w'_k}(x_k)+\left(1-\alpha_{w''_k}\right)b_k\delta_{w'_k,k}\right)^2|G_{d(k)}\right]\\
&\quad\leq (a_kC+a_kv_k+b_kD+b_k\sigma_k)^2.
\end{aligned}
\tag{10.7}
$$

As for the last term in the second " $\leq$ " of (10.2), since $G_{d(k)} \subset G_k$, it holds

$$
\begin{aligned}
&E\left[\left(2\alpha_{w''_k}a_k(x_k-y)^T\epsilon_{w_k,k}+2\left(1-\alpha_{w''_k}\right)b_k(x_k-y)^T\delta_{w'_k,k}\right)|G_{d(k)}\right]\\
&\quad=E\left[E\left[\left(2\alpha a_k(x_k-y)^T\epsilon_{w_k,k}+2\left(1-\alpha_{w''_k}\right)b_k(x_k-y)^T\delta_{w'_k,k}\right)|G_k\right]|G_{d(k)}\right]\\
&\quad=E\left[(x_k-y)^TE\left[\left(2\alpha_{w''_k}a_k\epsilon_{w_k,k}+2\left(1-\alpha_{w''_k}\right)b_k\delta_{w'_k,k}\right)|G_k\right]|G_{d(k)}\right]\\
&\quad\geq -E\left[||x_k-y||\left(||E[2a_k\epsilon_{w_k,k}|G_k]||+||E[2b_k\delta_{w'_k,k}|G_k]||\right)|G_{d(k)}\right]\\
&\quad\geq(-2a_k\mu_k-2b_k\tau_k)E\left[||x_k-y|||G_{d(k)}\right].
\end{aligned}
\tag{10.8}
$$

Substituting the preceding estimates (10.3)–(10.8) into (10.2) yields the desired estimate (7.1). □

Proof of Theorem 7.1

Proof Since Y is compact, f, g is convex functions, it follows that $Y^*(\alpha^*, \lambda)$ is nonempty, closed and convex. By Lemma 7.1, we obtain that for any $y^* \in Y^*(\alpha^*, \lambda)$

$$
\begin{aligned}
&E[\mathrm{dist}(\mathrm{x_{k+1}},\mathrm{Y}^*(\alpha^*,\lambda))^2|\mathrm{G_{d(k)}}]\leq \mathrm{E}[\mathrm{dist}(\mathrm{x_k},\mathrm{Y}^*(\alpha^*,\lambda))^2|\mathrm{G_{d(k)}}]\\
&\quad-\frac{2\alpha^*a_k}{m}\left(f(x_{d(k)})-f(y^*)\right)-\frac{2(1-\alpha^*)b_k}{n}\left(g(x_{d(k)})-g(y^*)\right)\\
&\quad+\left(2a_krC\left(b_R\beta_R^{k+1-d(k)}+mb_I\beta_I^{k+1-d(k)}\right)+2b_krD\left(b_R\beta_R^{k+1-d(k)}+nb_J\beta_J^{k+1-d(k)}\right)\right)\\
&\quad\times\max_{x,y\in Y}||x-y||+2(a_kC+b_kD)\sum_{\ell=d(k)}^{k-1}(a_\ell(C+v_\ell)+b_\ell(D+\sigma_\ell))\\
&\quad+(a_kC+a_kv_k+b_kD+b_k\sigma_k)^2+(2a_k\mu_k+2b_k\tau_k)E\left[\mathrm{dist}(\mathrm{x_k},\mathrm{Y}^*(\alpha^*,\lambda))|\mathrm{G_{d(k)}}\right].
\end{aligned}
\tag{10.9}
$$

Denote

$$\begin{aligned}\eta_k := &\Big(2a_k rC\Big(b_R\beta_R^{k+1-d(k)} + mb_I\beta_I^{k+1-d(k)}\Big) + 2b_k rD\Big(b_R\beta_R^{k+1-d(k)} + nb_J\beta_J^{k+1-d(k)}\Big)\Big)\\ &\times \max_{x,y\in Y}\|x-y\| + 2(a_kC + b_kD)\sum_{\ell=d(k)}^{k-1}(a_\ell(C+\nu_\ell) + b_\ell(D+\sigma_\ell))\\ &+ (a_kC + a_k\nu_k + b_kD + b_k\sigma_k)^2 + (2a_k\mu_k + 2b_k\tau_k)E[\mathrm{dist}(\mathrm{x_k}, \mathrm{Y}^*(\alpha^*,\lambda))].\end{aligned} \tag{10.10}$$

Taking expectations to (10.9) yields

$$\begin{aligned}&E\Big[\mathrm{dist}(\mathrm{x_{k+1}}, \mathrm{Y}^*(\alpha^*,\lambda))^2\Big]\\ &\quad\le E\Big[\mathrm{dist}(\mathrm{x_k}, \mathrm{Y}^*(\alpha^*,\lambda))^2\Big] - \frac{2\alpha^* a_k}{m}\big(E\big[f(x_{d(k)})\big] - f(y^*)\big)\\ &\qquad - \frac{2(1-\alpha^*)b_k}{n}\big(E\big[g(x_{d(k)})\big] - g(y^*)\big) + \eta_k\\ &\quad\le E\Big[\mathrm{dist}(\mathrm{x_k}, \mathrm{Y}^*(\alpha^*,\lambda))^2\Big] - \frac{2a_k}{m}\big(E\big[(\alpha^* f + \lambda(1-\alpha^*)g)(x_{d(k)})\big]\\ &\qquad - (\alpha^* f + \lambda(1-\alpha^*)g)(y^*)\big) + 2(1-\alpha^*)\left(\frac{b_k}{n} - \lambda\frac{a_k}{m}\right)\big(E\big[g\big(x_{d(k)}\big)\big] - g(y^*)\big) + \eta_k\\ &\quad\le E\Big[\mathrm{dist}(\mathrm{x_k}, \mathrm{Y}^*(\alpha^*,\lambda))^2\Big] - \frac{2a_k}{m}\big(E\big[(\alpha^* f + \lambda(1-\alpha^*)g)(x_{d(k)})\big]\\ &\qquad - (\alpha^* f + \lambda(1-\alpha^*)g)(y^*)\big) + 4M\left|\frac{b_k}{n} - \lambda\frac{a_k}{m}\right| + \eta_k.\end{aligned}$$

Following the same routine as in the proof of Theorem 4.3 in Ram et al. (2009), we easily know that for some non-negative integer sequence $\{d(k)\}$, it holds

$$\sum_{k=2}^{\infty}\eta_k < \infty.$$

In addition, in view of (7.2), the inequality

$$E\big[(\alpha^* f + \lambda(1-\alpha^*)g)(x_{d(k)})\big] \ge (\alpha^* f + \lambda(1-\alpha^*)g)(y^*)$$

and Lemma 1 in Bertsekas and Tsitsiklis (2000), we conclude that $E[\mathrm{dist}(\mathrm{x_k}, \mathrm{Y}^*(\alpha^*,\lambda))^2]$ converges to a non-negative scalar and

$$\sum_{k=2}^{\infty}\frac{2a_k}{m}\big(E\big[(\alpha^* f + \lambda(1-\alpha^*)g)(x_{d(k)})\big] - (\alpha^* f + \lambda(1-\alpha^*)g)(y^*)\big) < \infty, \tag{10.11}$$

which, together with the fact $\sum_{k=2}^{\infty} a_k = \infty$ for $\frac{2}{3} < p \le 1$, yields

$$\liminf_{k\to\infty} E[(\alpha^* f + \lambda(1-\alpha^*)g)(x_k)] = (\alpha^* f + \lambda(1-\alpha^*)g)(y^*). \tag{10.12}$$

Since f, g are continuous and Y is compact, from Fatou's lemma it follows

$$\begin{aligned}E\Big[\liminf_{k\to\infty}(\alpha^* f + \lambda(1-\alpha^*)g)(x_k)\Big] &\le \liminf_{k\to\infty} E[(\alpha^* f + \lambda(1-\alpha^*)g)(x_k)]\\ &= (\alpha^* f + \lambda(1-\alpha^*)g)(y^*),\end{aligned} \tag{10.13}$$

which implies that

$$\liminf_{k\to\infty}(\alpha^* f + \lambda(1-\alpha^*)g)(x_k) = (\alpha^* f + \lambda(1-\alpha^*)g)(y^*)$$

with probability 1, i.e. the first result of (7.4) holds. Using again the continuity of f, g and the compactness of Y, we know that

$$\liminf_{k\to\infty} \mathrm{dist}(\mathrm{x_k}, \mathrm{Y}^*(\alpha^*, \lambda)) = 0$$

with probability 1, i.e. the second result of (7.4) holds.

Now we aim to prove (7.5). Since $\liminf_{k\to\infty} \mathrm{dist}(\mathrm{x_k}, \mathrm{Y}^*(\alpha^*, \lambda)) = 0$ with probability 1, there exists a subsequence of $\{\mathrm{dist}(\mathrm{x_k}, \mathrm{Y}^*(\alpha^*, \lambda))^2\}$, which we denote by $\{\mathrm{dist}(\mathrm{x}_{\mathrm{k}_\ell}, \mathrm{Y}^*(\alpha^*, \lambda))^2\}$, such that $\lim_{k_\ell\to\infty} \mathrm{dist}(\mathrm{x}_{\mathrm{k}_\ell}, \mathrm{Y}^*(\alpha^*, \lambda))^2 = 0$ with probability 1. For the set Y is bounded, we know the sequence $\{\mathrm{dist}(\mathrm{x}_{\mathrm{k}_\ell}, \mathrm{Y}^*(\alpha^*, \lambda))^2\}$ is bounded. By the dominated convergence theorem, we have

$$\lim_{k_\ell\to\infty} E\left[\mathrm{dist}(\mathrm{x}_{\mathrm{k}_\ell}, \mathrm{Y}^*(\alpha^*, \lambda))^2\right] = E\left[\lim_{k_\ell\to\infty} \mathrm{dist}(\mathrm{x}_{\mathrm{k}_\ell}, \mathrm{Y}^*(\alpha^*, \lambda))^2\right] = 0.$$

Since we already obtain that $E[\mathrm{dist}(\mathrm{x_k}, \mathrm{Y}^*(\alpha^*, \lambda))^2]$ converges to a non-negative scalar, then it has to converge to 0, i.e.

$$\lim_{k\to\infty} E\left[\mathrm{dist}\left(\mathrm{x_k}, \mathrm{Y}^*(\alpha^*, \lambda)^2\right] = 0,$$

which completes the proof. □

Experiments for Sect. 8.5

The weight α follows a Markov chain in a state $R = \{\alpha_1, \alpha_2, \alpha_3\}$ and in all cases, the transition matrix for the Markov chain for α is given by, $k = 0, 1, 2, \ldots$,

$$P_R(k) = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{pmatrix},$$

which equals P_I in Fig. 2. We let $\lambda = 1$. The two step sizes are given by

$$a_k = \frac{1}{k+1} \quad \text{and} \quad b_k = \frac{\lambda n}{m} a_k$$

so that the λ condition (7.2) is satisfied. In the situations with noises, we add the Gauss noises with mean and variance (0, 25), as before.

Experiment 1 $m = 5, n = 5$. The two transition matrices $P_I(k)$ and $P_J(k)$ are given by, $k = 0, 1, 2, \ldots$,

$$P_I(k) = \begin{pmatrix} 4/5 & 1/5 & 0 & 0 & 0 \\ 1/5 & 3/5 & 1/5 & 0 & 0 \\ 0 & 1/5 & 3/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 3/5 & 1/5 \\ 0 & 0 & 0 & 1/5 & 4/5 \end{pmatrix},$$

$$P_J(k) = \begin{pmatrix} 3/5 & 1/5 & 0 & 0 & 1/5 \\ 1/5 & 3/5 & 1/5 & 0 & 0 \\ 0 & 1/5 & 3/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 3/5 & 1/5 \\ 1/5 & 0 & 0 & 1/5 & 3/5 \end{pmatrix}.$$

We let $R = \{0.25, 0.5, 0.75\}$. Thus, $\alpha^* = 0.5$ and $Y^* = Y^*(\alpha^*, \lambda) = \{1\}$, since $\lambda = 1$. Figure 8 reports the result, which shows that the price process $\{x_k\}$ converges to the equilibrium price 1, with or without noises. Next we increase the number of buyers from $n = 5$ to $n = 10$ while m remains at 5.

Experiment 2 $m = 5, n = 10$. The two transition matrices $P_I(k)$ and $P_J(k)$ are given by, $k = 0, 1, 2, \ldots,$

$$P_I(k) = \begin{pmatrix} 4/5 & 1/5 & 0 & 0 & 0 \\ 1/5 & 3/5 & 1/5 & 0 & 0 \\ 0 & 1/5 & 3/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 3/5 & 1/5 \\ 0 & 0 & 0 & 1/5 & 4/5 \end{pmatrix},$$

$$P_J(k) = \begin{pmatrix} 8/10 & 1/10 & & & 1/10 \\ 1/10 & 8/10 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1/10 \\ 1/10 & & & 1/10 & 8/10 \end{pmatrix}_{10\times 10}.$$

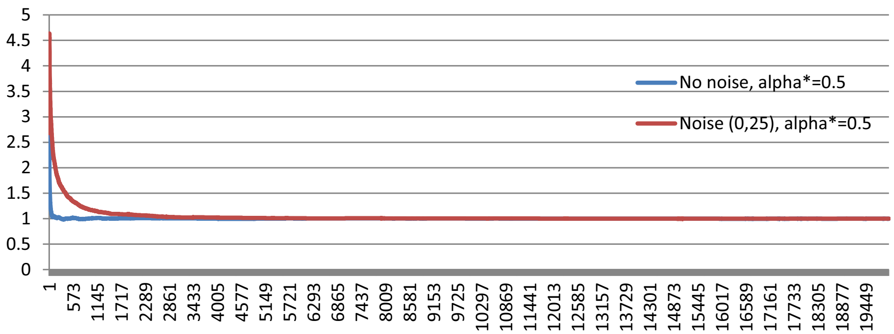

Fig. 8 Average $\{x_k\}$ of 100 samples for m = 5 and n = 5. x_0 randomly generated in [1, 5] and iterations from 1 to 20,000. *Note*: equilibrium price = 1 when $\alpha^* = 0.5$

We select two different states for αs, $R = \{0.25, 0.5, 0.75\}$ and $R1 = \{0.25, 0.75, 1\}$, with different average weights $\alpha^* = \frac{1}{2}$ and $\alpha_1^* = \frac{2}{3}$. Thus, $Y^* = Y^*(\alpha^*, \lambda) = \{1.8028\}$ and $Y^*(\alpha_1^*, \lambda) = \{1.2748\}$. Clearly, $Y^* \neq Y^*(\alpha_1^*, \lambda)$. Figure 9 presents the results, which confirm what has been shown in Theorem 7.1. Note that the fundamentals remain the same. But the price process converges to the price 1.2748, purely due to a change in the average weight from α^* to α_1^*. The interesting part is that what matters is the average α^*. This implies that if we change $R1$ to $R1' = \{0.1, 0.9, 1\}$, then our experiments will also converge to the same price 1.2748. Next we increase the number of buyers further from $n = 10$ to $n = 20$ while m stays put at 5.

Experiment 3 $m = 5, n = 20$. The two transition matrices $P_I(k)$ and $P_J(k)$ are given by, $k = 0, 1, 2, \ldots,$

$$P_I(k) = \begin{pmatrix} 4/5 & 1/5 & 0 & 0 & 0 \\ 1/5 & 3/5 & 1/5 & 0 & 0 \\ 0 & 1/5 & 3/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 3/5 & 1/5 \\ 0 & 0 & 0 & 1/5 & 4/5 \end{pmatrix},$$

$$P_J(k) = \begin{pmatrix} 18/20 & 1/20 & & & 1/20 \\ 1/20 & 18/20 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1/20 \\ 1/20 & & & 1/20 & 18/20 \end{pmatrix}_{20\times 20}.$$

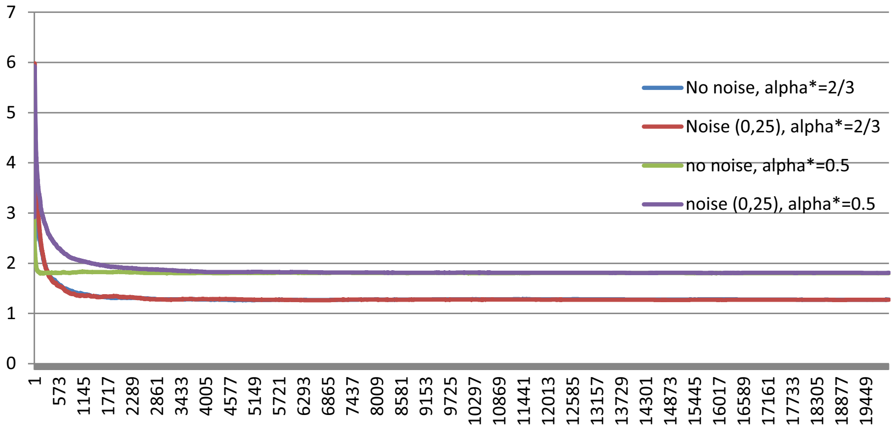

Fig. 9 Average $\{x_k\}$ of 100 samples for $m = 5$ and $n = 10$. x_0 randomly generated in [1, 5] and iterations from 1 to 20,000. *Note*: equilibrium price = 1.8 when α^* is 0.5. For $\alpha^* = 2/3$, the process converges to 1.27

We also consider two different states $R = \{0.25, 0.5, 0.75\}$ and $R2 = \{0.4, 0.8, 1\}$. Then $\alpha^* = 0.5$ and $\alpha_2^* = 2.2/3$. So, $Y^* = \{3.3912\}$ and $Y^*(\alpha_2^*, \lambda) = \{2.0449\}$. Figure 10 presents the experimental results. Once again, with a higher average weight α_2^* than α^*, the price process converges to a price that is substantially lower than the equilibrium price of the original economy. Next we keep $n = 5$ as in Experiment 1 while increase the number of sellers from $m = 5$ to $m = 10$. One can expect that the equilibrium price is lower than 1 because there are more sellers.

Experiment 4 $m = 10, n = 5$. The two transition matrices $P_I(k)$ and $P_J(k)$ are given by, $k = 0, 1, 2, \ldots,$

$$P_I(k) = \begin{pmatrix} 9/10 & 1/10 & & & \\ 1/10 & 8/10 & \ddots & & \\ & \ddots & \ddots & 1/10 & \\ & & 1/10 & 8/10 & 1/10 \\ & & & 1/10 & 9/10 \end{pmatrix}_{10\times 10},$$

$$P_J(k) = \begin{pmatrix} 3/5 & 1/5 & 0 & 0 & 1/5 \\ 1/5 & 3/5 & 1/5 & 0 & 0 \\ 0 & 1/5 & 3/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 3/5 & 1/5 \\ 1/5 & 0 & 0 & 1/5 & 3/5 \end{pmatrix}.$$

We set $R = \{0.25, 0.5, 0.75\}$ and $R3 = \{0, 0.25, 0.75\}$ in our experiments. Thus, $\alpha^* = 0.5$ and $\alpha_3^* = \frac{1}{3}$. So, $Y^* = \{0.5547\}$ and $Y^*(\alpha_3^*, \lambda) = \{0.7845\}$. The experimental results are reported in Fig. 11, which confirms our theoretical result in

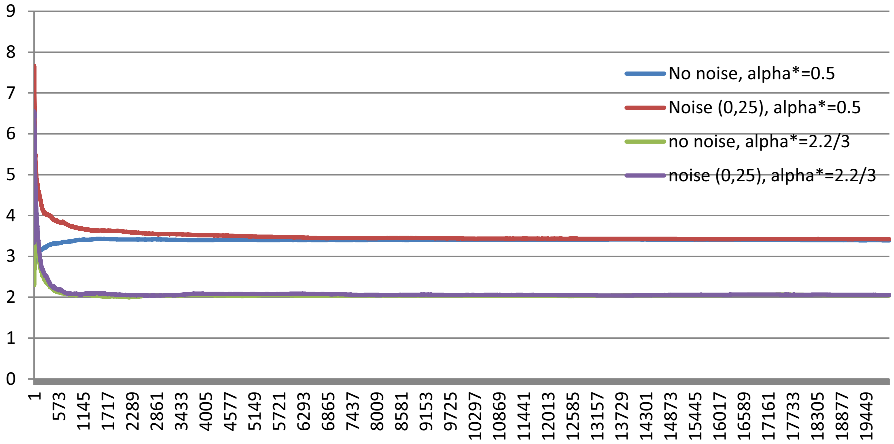

Fig. 10 Average $\{x_k\}$ of 100 samples for $m = 5$ and $n = 20$. x_0 randomly generated in [1, 5] and iterations from 1 to 20,000. *Note*: equilibrium $= 3.39$ for $\alpha^* = 0.5$. For $\alpha^* = 2.2/3$, the process converges to 2.04

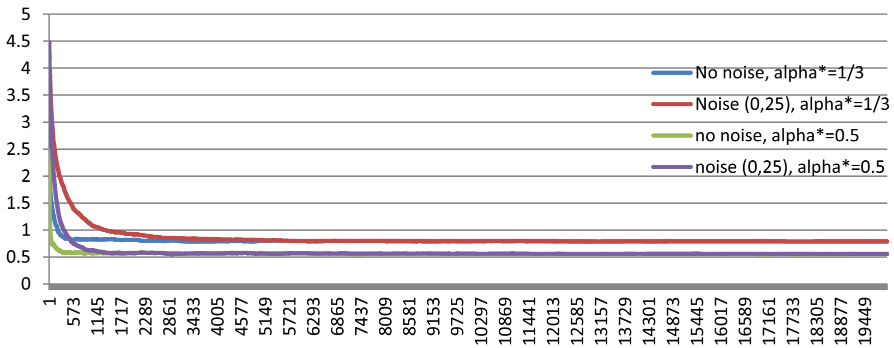

Fig. 11 Average $\{x_k\}$ of 100 samples for m = 10 and n = 5. x_0 randomly generated in [1, 5] and iterations from 1 to 20,000. *Note*: equilibrium = 0.55 when α^* equals 0.5 For $\alpha^* = 1/3$, the process converges to 0.78

Theorem 7.1. Note that when α^* is lower from $\frac{1}{2}$ to $\frac{1}{3}$, the price process converges to a higher price 0.7845 than the equilibrium price 0.5547 of the original economy, higher by more than 41 %. Next we increase the number of sellers from $m = 10$ to $m = 20$. The equilibrium price in Y^* will be even lower, as expected.

Experiment 5 $m = 20, n = 5$. The two transition matrices $P_I(k)$ and $P_J(k)$ are given by, $k = 0, 1, 2, \ldots,$

$$P_I(k) = \begin{pmatrix} 19/20 & 1/20 & & & \\ 1/20 & 18/20 & \ddots & & \\ & \ddots & \ddots & 1/20 & \\ & & 1/20 & 18/20 & 1/20 \\ & & & 1/20 & 19/20 \end{pmatrix}_{20\times 20},$$

$$P_J(k) = \begin{pmatrix} 3/5 & 1/5 & 0 & 0 & 1/5 \\ 1/5 & 3/5 & 1/5 & 0 & 0 \\ 0 & 1/5 & 3/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 3/5 & 1/5 \\ 1/5 & 0 & 0 & 1/5 & 3/5 \end{pmatrix}.$$

We set $R = \{0.25, 0.5, 0.75\}$ and $R4 = \{0, 0.2, 0.4\}$ in our experiments. Thus, $\alpha^* = 0.5$ and $\alpha_4^* = 0.2$. So, $Y^* = \{0.2949\}$ and $Y^*(\alpha_4^*, \lambda) = \{0.5898\}$. The experimental results are reported in Fig. 12. When the average weight α^* moves lower from 0.5 to 0.2, the price process converges to a higher price 0.5898, which is twice as much as the equilibrium price of the original economy because $\sqrt{\frac{\lambda(1-\alpha_4^*)}{\alpha_4^*}} = 2$. In summary, we have done 18 experiments each of which has shown how a change in α may affect the convergence of the price process $\{x_k\}$ of an α-double auction. These

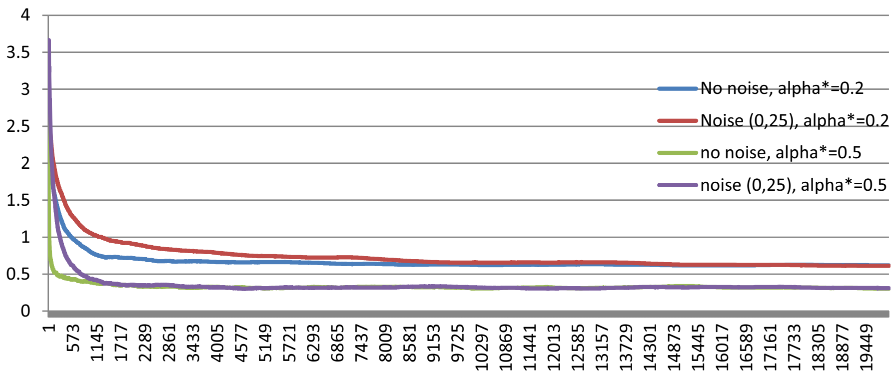

Fig. 12 Average $\{x_k\}$ of 100 samples for $m = 20$ and $n = 5$. x_0 randomly generated in [1, 5] and iterations from 1 to 20,000. *Note*: equilibrium price = 0.29 for $\alpha^* = 0.5$. For $\alpha^* = 1/5$, the process converges to 0.59

experiments provide solid evidence a double auction implemented in a real exchange market may indeed contribute to the excess volatility.

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